Chapter 7: Voting Systems

Section 7.1: Voting Methods

Every couple of years or so, voters go to the polls to cast ballots for their choices for mayor, governor, senator, president, etc. Then the election officials count the ballots and declare a winner. But how do the election officials determine who the winner is. If there are only two candidates, then there is no problem figuring out the winner. The candidate with more than 50% of the votes wins. This is known as the majority. So the candidate with the majority of the votes is the winner.

Majority Rule: This concept means that the candidate (choice) receiving more than 50% of the vote is the winner.

But what happens if there are three candidates, and no one receives the majority? That depends on where you live. Some places decide that the person with the most votes wins, even if they don't have a majority. There are problems with this, in that someone could be liked by 35% of the people, but is disliked by 65% of the people. So you have a winner that the majority doesn't like. Other places conduct runoff elections where the top two candidates have to run again, and then the winner is chosen from the runoff election. There are some problems with this method. First, it is very costly for the candidates and the election office to hold a second election. Second, you don't know if you will have the same voters voting in the second election, and so the preferences of the voters in the first election may not be taken into account.

So what can be done to have a better election that has someone liked by more voters yet doesn't require a runoff election? A ballot method that can fix this problem is known as a preference ballot.

Preference Ballots: Ballots in which voters choose not only their favorite candidate, but they actually order all of the candidates from their most favorite down to their least favorite.

Note: Preference Ballots are transitive: If a voter prefers choice A to choice B and also prefers choice B to choice C, then the voter must prefer choice A to choice C.

To understand how a preference ballot works and how to determine the winner, we will look at an example.

Example 7.1.1: Preference Ballot for the Candy Election

Suppose an election is held to determine which bag of candy will be opened. The choices (candidates) are Hershey's Miniatures (M), Nestle Crunch (C), and Mars' Snickers (S). Each voter is asked to fill in the following ballot, by marking their first, second, and third place choices.

Figure 7.1.1: Preference Ballot for the Candy Election

Candy	Preference
Crunch	__________
Miniatures	__________
Snickers	__________

Each voter fills out the above ballot with their preferences, and what follows is the results of the election.

Table 7.1.2: Ballots Cast for the Candy Election

Voter	Anne	Bob	Chloe	Dylan	Eli	Fred
1st choice	C	M	C	M	S	S
2nd choice	S	S	M	C	M	M
3rd choice	M	C	S	S	C	C

Voter	George	Hiza	Isha	Jacy	Kalb	Lan
1st choice	S	S	S	M	C	M
2nd choice	M	M	M	C	M	C
3rd choice	C	C	C	S	S	S

Voter	Makya	Nadira	Ochen	Paki	Quinn	Riley
1st choice	S	S	C	C	S	S
2nd choice	M	M	M	M	M	M
3rd choice	C	C	S	S	C	C

Now we must count the ballots. It isn't as simple as just counting how many voters like each candidate. You have to look at how many liked the candidate in first-place, second place, and third place. So there needs to be a better way to organize the results. This is known as a preference schedule.

Preference Schedule: A table used to organize the results of all the preference ballots in an election.

Example 7.1.2: Preference Schedule for the Candy Election

Using the ballots from Example 7.1.1, we can count how many people liked each ordering. Looking at Table 7.1.2, you may notice that three voters (Dylan, Jacy, and Lan) had the order M, then C, then S. Bob is the only voter with the order M, then S, then C. Chloe, Kalb, Ochen, and Paki had the order C, M, S. Anne is the only voter who voted C, S, M. All the other 9 voters selected the order S, M, C. Notice, no voter liked the order S, C, M. We can summarize this information in a table, called the preference schedule.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

Methods of Counting Ballots:

Now that we have organized the ballots, how do we determine the winner? There are several different methods that can be used. The easiest, and most familiar, is the Plurality Method.

Plurality Method: The candidate with the most first-place votes wins the election.

Example 7.1.3: The Winner of the Candy Election—Plurality Method

Using the preference schedule in Table 7.1.3, find the winner using the Plurality Method.

From the preference schedule you can see that four (3 + 1) people choose Hershey's Miniatures as their first choice, five (4 + 1) picked Nestle Crunch as their first choice, and nine picked Snickers as their first choice. So Snickers wins with the most first-place votes, although Snickers does not have the majority of first-place votes.

There is a problem with the Plurality Method. Notice that nine people picked Snickers as their first choice, yet seven chose it as their third choice. Thus, nine people may be happy if the Snickers bag is opened, but seven people will not be happy at all. So let's look at another way to determine the winner.

The Borda Count Method (Point System): Each place on a preference ballot is assigned points. Last place receives one point, next to last place receives two points, and so on. Thus, if there are *N* candidates, then first-place receives *N* points. Now, multiply the point value for each place by the number of voters at the top of the column to find the points each candidate wins in a column. Lastly, total up all the points for each candidate. The candidate with the most points wins.

Example 7.1.4: The Winner of the Candy Election—Borda Count Method

Using the preference schedule in Table 7.1.3, find the winner using the Borda Count Method.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

The third choice receives one point, second choice receives two points, and first choice receives three points. There were three voters who chose the order M, C, S. So M receives 3*3 = 9 points for the first-place, C receives 3*2 = 6 points, and S receives 3*1 = 3 points for those ballots. The same process is conducted for the other columns. The table below summarizes the points that each candy received.

Table 7.1.4: Preference Schedule of the Candy Election with Borda Count Points

Number of voters	3	1	4	1	9
1st choice	M 9	M 3	C 12	C 3	S 27
2nd choice	C 6	S 2	M 8	S 2	M 18
3rd choice	S 3	C 1	S 4	M 1	C 9

Adding up these points gives,
M = 9 + 3 + 8 + 1 + 18 = 39
C = 6 + 1 + 12 + 3 + 9 = 31
S = 3 + 2 + 4 + 2 + 27 = 38

Thus, Hershey’s Miniatures wins using the Borda Count Method.

So who is the winner? With one method Snicker's wins and with another method Hershey's Miniatures wins. The problem is that it all depends on which method you use. Therefore, you need to decide which method to use before you run the election.

The Plurality with Elimination Method (Sequential Runoffs): Eliminate the candidate with the least amount of 1st place votes and re-distribute their votes amongst the other candidates. Repeat this process until you find a winner. *Note: At any time during this process if a candidate has a majority of first-place votes, then that candidate is the winner.*

Example 7.1.5: The Winner of the Candy Election—Plurality with Elimination Method

Using the preference schedule in Table 7.1.3, find the winner using the Plurality with Elimination Method.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

This isn't the most exciting example, since there are only three candidates, but the process is the same whether there are three or many more. So look at how many first-place votes there are. M has $3+1=4$, C has $4+1=5$, and S has 9. So M is eliminated from the preference schedule.

Table 7.1.5: Preference Schedule for the Candy Election with M Eliminated

Number of voters	3	1	4	1	9
1st choice	~~M~~	~~M~~	C	C	S
2nd choice	C	S	~~M~~	S	~~M~~
3rd choice	S	C	S	~~M~~	C

So the preference schedule becomes:

Table 7.1.6: Preference Schedule for the Candy Election with M Eliminated

Number of voters	3	1	4	1	9
1st choice	C	S	C	C	S
2nd choice	S	C	S	S	C

And then we can condense it down to:

Table 7.1.7: Preference Schedule for the Candy Election Condensed

Number of voters	8	10
1st choice	C	S
2nd choice	S	C

So C has eight first-place votes, and S has 10. So S wins.

The Method of Pairwise Comparisons: Compare each candidate to the other candidates in one-on-one match-ups. Give the winner of each pairwise comparison a point. The candidate with the most points wins.

Example 7.1.6: The Winner of the Candy Election—Pairwise Comparisons Method

Using the preference schedule in Table 7.1.3, find the winner using the Pairwise Comparisons Method.

Table 7.1.3: Preference Schedule for the Candy Election

Number of voters	3	1	4	1	9
1st choice	M	M	C	C	S
2nd choice	C	S	M	S	M
3rd choice	S	C	S	M	C

If you only have an election between M and C (the first one-on-one match-up), then M wins the three votes in the first column, the one vote in the second column, and the nine votes in the last column. That means that M has thirteen votes while C has five. So M wins when compared to C. M gets one point.

If you only compare M and S (the next one-on-one match-up), then M wins the first three votes in column one, the next one vote in column two, and the four votes in column three. M has eight votes and S has 10 votes. So S wins compared to M, and S gets one point.

Comparing C to S, C wins the three votes in column one, the four votes in column three, and one vote in column four. C has eight votes while S has 10 votes. So S wins compared to C, and S gets one point.

To summarize, M has one point, and S has two points. Thus, S wins the election using the Method of Pairwise Comparisons.

Table 7.1.8: Summary of One-on-One Match-Ups for the Candy Election

Match-Up 1	**Match-Up 2**	**Match-Up 3**
M vs. C	M vs. S	S vs. C
13 to 5	8 to 10	10 to 8
Winner of Match-Up 1: M	Winner of Match-Up 2: S	Winner of Match-Up 3: S

M: 1
S: 2
C: 0
Thus, S wins the election.

Note: If any one given match-up ends in a tie, then both candidates receive ½ point each for that match-up.

The problem with this method is that many overall elections (not just the one-on-one match-ups) will end in a tie, so you need to have a tie-breaker method designated before beginning the tabulation of the ballots. Another problem is that if there are more than three candidates, the number of pairwise comparisons that need to be analyzed becomes unwieldy. So, how many pairwise comparisons are there?

In Example 7.1.6, there were three one-on-one comparisons when there were three candidates. You may think that means the number of pairwise comparisons is the same as the number of candidates, but that is not correct. Let's see if we can come up with a formula for the number of candidates. Suppose you have four candidates called A, B, C, and D. A is to be matched up with B, C, and D (three comparisons). B is to be compared with C and D, but has already been compared with A (two comparisons). C needs to be compared with D, but has already been compared with A and B (one more comparison). Therefore, the total number of one-on-one match-ups is $3+2+1=6$ comparisons that need to be made with four candidates. What about five or six or more candidates? Looking at five candidates, the first candidate needs to be matched-up with four other candidates, the second candidate needs to be matched-up with three other candidates, the third candidate needs to be matched-up with two other candidates, and the fourth candidate needs to only be matched-up with the last candidate for one more match-up. Thus, the total is $4+3+2+1=10$ pairwise comparisons when there are five candidates.

Now, for six candidates, you would have $5+4+3+2+1=15$ pairwise comparisons to do. Continuing this pattern, if you have N candidates then there are $(N-1)+(N-2)+\cdots+3+2+1$ pairwise comparisons. For small numbers of candidates, it isn't hard to add these numbers up, but for large numbers of candidates there is a shortcut for adding the numbers together. It turns out that the following formula is true:

$(N-1)+(N-2)+\cdots+3+2+1=\frac{N(N-1)}{2}$. Thus, for 10 candidates, there are

$9+8+7+6+5+4+3+2+1=\frac{10(10-1)}{2}=\frac{10(9)}{2}=45$ pairwise comparisons. So you can see that in this method, the number of pairwise comparisons to do can get large quite quickly.

Now that we have reviewed four different voting methods, how do you decide which method to use? One question to ask is which method is the fairest? Unfortunately, there is no completely fair method. This is based on Arrow's Impossibility Theorem.

Arrow's Impossibility Theorem: No voting system can satisfy all four fairness criteria in all cases.

This brings up the question, what are the four fairness criteria? They are guidelines that people use to help decide which voting method would be best to use under certain circumstances. They are the Majority Criterion, Condorcet Criterion, Monotonicity Criterion, and Independence of Irrelevant Alternatives Criterion.

Fairness Criteria:
The Majority Criterion (Criterion 1): If a candidate receives a majority of the 1st-place votes in an election, then that candidate should be the winner of the election.
The Condorcet Criterion (Criterion 2): If there is a candidate that in a head-to-head comparison is preferred by the voters over every other candidate, then that candidate should be the winner of the election. This candidate is known as the Condorcet candidate.
The Monotonicity Criterion (Criterion 3): If candidate X is a winner of an election and, in a re-election, the only changes in the ballots are changes that favor X, then X should remain a winner of the election.
The Independence of Irrelevant Alternatives Criterion (Criterion 4): If candidate X is a winner of an election and one (or more) of the other candidates is removed and the ballots recounted, then X should still be a winner of the election.

Example 7.1.7: Condorcet Criterion Violated

Suppose you have a vacation club trying to figure out where it wants to spend next year's vacation. The choices are Hawaii (H), Anaheim (A), or Orlando (O). The preference schedule for this election is shown below in Table 7.1.9.

Table 7.1.9: Preference Schedule of Vacation Election

Number of voters	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Using the Plurality Method, A has four first-place votes, O has three first-place votes, and H has three first-place votes. So, Anaheim is the winner. However, if you use the Method of Pairwise Comparisons, A beats O (A has seven while O has three), H beats A (H has six while A has four), and H beats O (H has six while O has four). Thus, Hawaii wins all pairwise comparisons against the other candidates, and would win the election. In fact Hawaii is the Condorcet candidate. However, the Plurality Method declared Anaheim the winner, so the Plurality Method violated the Condorcet Criterion.

Example 7.1.8: Monotonicity Criterion Violated

Suppose you have a voting system for a mayor. The resulting preference schedule for this election is shown below in Table 7.1.10.

Table 7.1.10: Preference Schedule of Mayoral Election

Number of voters	37	22	12	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

Using the Plurality with Elimination Method, Adams has 37 first-place votes, Brown has 34, and Carter has 29, so Carter would be eliminated. Carter's votes go to Adams, and Adams wins. Suppose that the results were announced, but then the election officials accidentally destroyed the ballots before they could be certified, so the election must be held again. Wanting to "jump on the bandwagon," 10 of the voters who had originally voted in the order Brown, Adams, Carter; change their vote to the order of Adams, Brown, Carter. No other voting changes are made. Thus, the only voting changes are in favor of Adams. The new preference schedule is shown below in Table 7.1.11.

Table 7.1.11: Preference Schedule of Mayoral Re-election

Number of voters	47	22	2	29
1st choice	Adams	Brown	Brown	Carter
2nd choice	Brown	Carter	Adams	Adams
3rd choice	Carter	Adams	Carter	Brown

Now using the Plurality with Elimination Method, Adams has 47 first-place votes, Brown has 24, and Carter has 29. This time, Brown is eliminated first instead of Carter. Two of Brown's votes go to Adams and 22 of Brown's votes go to Carter. Now, Adams has 47 + 2 = 49 votes and Carter has 29 + 22 = 51 votes. Carter wins the election. This doesn't make sense since Adams had won the election before, and the only changes that were made to the ballots were in favor of Adams. However, Adams doesn't win the re-election. The reason that this happened is that there was a difference in who was eliminated first, and that caused a difference in how the votes are re-distributed. In this example, the Plurality with Elimination Method violates the Monotonicity Criterion.

Example 7.1.9: Majority Criterion Violated

Suppose a group is planning to have a conference in one of four Arizona cities: Flagstaff, Phoenix, Tucson, or Yuma. The votes for where to hold the conference are summarized in the preference schedule shown below in Table 7.1.12.

Table 7.1.12: Preference Schedule for Conference City

Number of voters	51	25	10	14
1st choice	Flagstaff	Phoenix	Yuma	Tucson
2nd choice	Phoenix	Yuma	Phoenix	Phoenix
3rd choice	Tucson	Tucson	Tucson	Yuma
4th choice	Yuma	Flagstaff	Flagstaff	Flagstaff

If we use the Borda Count Method to determine the winner then the number of Borda points that each candidate receives are shown in Table 7.1.13.

Table 7.1.13: Preference Schedule for Conference City with Borda Points

Number of voters	51	25	10	14
1st choice **4 points**	Flagstaff 204	Phoenix 100	Yuma 40	Tucson 56
2nd choice **3 points**	Phoenix 153	Yuma 75	Phoenix 30	Phoenix 42
3rd choice **2 points**	Tucson 102	Tucson 50	Tucson 20	Yuma 28
4th choice **1 point**	Yuma 51	Flagstaff 25	Flagstaff 10	Flagstaff 14

The totals of all the Borda points for each city are:

Flagstaff: $204+25+10+14=253$ points

Phoenix: $153+100+30+42=325$ points

Yuma: $51+75+40+28=194$ points

Tucson: $102+50+20+56=228$ points

Phoenix wins using the Borda Count Method. However, notice that Flagstaff actually has the majority of first-place votes. There are 100 voters total and 51 voters voted for Flagstaff in first place (51/100 = 51% or a majority of the first-place votes). So, Flagstaff should have won based on the Majority Criterion. This shows how the Borda Count Method can violate the Majority Criterion.

Example 7.1.10: Independence of Irrelevant Alternatives Criterion Violated

A committee is trying to award a scholarship to one of four students: Anna (A), Brian (B), Carlos (C), and Dmitri (D). The votes are shown below.

Table 7.1.14: Preference Schedule for Scholarship

Number of voters	5	5	6	4
1st choice	D	A	C	B
2nd choice	A	C	B	D
3rd choice	C	B	D	A
4th choice	B	D	A	C

Using the Method of Pairwise Comparisons:
A vs B: 10 votes to 10 votes, A gets ½ point and B gets ½ point
A vs C: 14 votes to 6 votes, A gets 1 point
A vs D: 5 votes to 15 votes, D gets 1 point
B vs C: 4 votes to 16 votes, C gets 1 point
B vs D: 15 votes to 5 votes, B gets 1 point
C vs D: 11 votes to 9 votes, C gets 1 point

So A has 1½ points, B has 1 point, C has 2 points, and D has 1 point. So Carlos is awarded the scholarship.

Now suppose it turns out that Dmitri didn't qualify for the scholarship after all. Though it should make no difference, the committee decides to recount the vote. The preference schedule without Dmitri is below.

Table 7.1.15: Preference Schedule for Scholarship with Dmitri Removed

Number of voters	10	6	4
1st choice	A	C	B
2nd choice	C	B	A
3rd choice	B	A	C

Using the Method of Pairwise Comparisons:
A vs B: 10 votes to 10 votes, A gets ½ point and B gets ½ point
A vs C: 14 votes to 6 votes, A gets 1 point
B vs C: 4 votes to 16 votes, C gets 1 point

So A has 1½ points, B has ½ point, and C has 1 point. Now Anna is awarded the scholarship instead of Carlos. This is an example of The Method of Pairwise Comparisons violating the Independence of Irrelevant Alternatives Criterion.

In summary, every one of the fairness criteria can possibly be violated by at least one of the voting methods as shown in Table 7.1.16. However, keep in mind that this does not mean that the voting method in question will violate a criterion in every election. It is just important to know that these violations are possible.

Table 7.1.16: Summary of Violations of Fairness Criteria

	Plurality	Borda Count	Plurality with Elimination	Pairwise Comparisons
Majority Criterion	*	Violation Possible	*	*
Condorcet Criterion	Violation Possible	Violation Possible	Violation Possible	*
Monotonicity Criterion	*	*	Violation Possible	*
Independence of Irrelevant Alternatives Criterion	Violation Possible	Violation Possible	Violation Possible	Violation Possible

* *The indicated voting method does not violate the indicated criterion in any election.*

Insincere Voting:
This is when a voter will not vote for whom they most prefer because they are afraid that the person they are voting for won't win, and they really don't want another candidate to win. So, they may vote for the person whom they think has the best chance of winning over the person they don't want to win. This happens often when there is a third party

candidate running. As an example, if a Democrat, a Republican, and a Libertarian are all running in the same race, and you happen to prefer the Libertarian candidate. However, you are afraid that the Democratic candidate will win if you vote for the Libertarian candidate, so instead you vote for the Republican candidate. You have voted insincerely to your true preference.

Approval Voting:
Since there is no completely fair voting method, people have been trying to come up with new methods over the years. One idea is to have the voters decide whether they approve or disapprove of candidates in an election. This way, the voter can decide that they would be happy with some of the candidates, but would not be happy with the other ones. A possible ballot in this situation is shown in Table 7.1.17:

Table 7.1.17: Approval Voting Ballot

Candidate	Approve	Disapprove
Smith	X	
Baker		X
James		X
Paulsen	X	

This voter would approve of Smith or Paulsen but would not approve of Baker or James. In this type of election, the candidate with the most approval votes wins the election.

One issue with approval voting is that it tends to elect the least disliked candidate instead of the best candidate. Another issue is that it can result in insincere voting as described above.
As a reminder, there is no perfect voting method. Arrow proved that there never will be one. So, make sure that you determine the method of voting that you will use before you conduct an election.

Section 7.2: Weighted Voting

Voting Power:
There are some types of elections where the voters do not all have the same amount of power. This happens often in the business world where the power that a voter possesses may be based on how many shares of stock he/she owns. In this situation, one voter may control the equivalent of 100 votes where other voters only control 15 or 10 or fewer votes. Therefore, the amount of power that each voter possesses is different. Another example is in how the President of the United States is elected. When a person goes to the

polls and casts a vote for President, he or she is actually electing who will go to the Electoral College and represent that state by casting the actual vote for President. Each state has a certain number of Electoral College votes, which is determined by the number of Senators and number of Representatives in Congress. Some states have more Electoral College votes than others, so some states have more power than others. How do we determine the power that each state possesses?

To figure out power, we need to first define some concepts of a weighted voting system. The individuals or entities that vote are called **players**. The notation for the players is $P_1, P_2, P_3, \ldots, P_N$, where N is the number of players. Each player controls a certain number of votes, which are called the **weight** of that player. The notation for the weights is $w_1, w_2, w_3, \ldots, w_N$, where w_1 is the weight of P_1, w_2 is the weight of P_2, etc. In order for a motion to pass, it must have a minimum number of votes. This minimum is known as the **quota**. The notation for quota is q. The quota must be over half the total weights and cannot be more than total weight. In other words,

$$\frac{w_1 + w_2 + w_3 + \cdots w_N}{2} < q \leq w_1 + w_2 + w_3 + \cdots + w_N$$

The way to denote a weighted voting system is $[q : w_1, w_2, w_3, \ldots, w_N]$.

Example 7.2.1: Weighted Voting System

A company has 5 shareholders. Ms. Lee has 30% ownership, Ms. Miller has 25%, Mr. Matic has 22% ownership, Ms. Pierce has 14%, and Mr. Hamilton has 9%. There is a motion to decide where best to invest their savings. The company's by-laws define the quota as 58%. What does this voting system look like?

Treating the percentages of ownership as the votes, the system looks like:
$[58 : 30, 25, 22, 14, 9]$

Example 7.2.2: Valid Weighted Voting System

Which of the following are valid weighted voting systems?

a. $[8 : 5, 4, 4, 3, 2]$

The quota is 8 in this example. The total weight is $5+4+4+3+2=18$. Half of 18 is 9, so the quota must be $9 < q \leq 18$. Since the quota is 8, and 8 is not more than 9, this system is not valid.

b. $[16:6,5,3,1]$

The quota is 16 in this example. The total weight is $6+5+3+1=15$. Half of 15 is 7.5, so the quota must be $7.5 < q \leq 15$. Since the quota is 16, and 16 is more than 15, this system is not valid.

c. $[9:5,4,4,3,1]$

The quota is 9 in this example. The total weight is $5+4+4+3+1=17$. Half of 17 is 8.5, so the quota must be $8.5 < q \leq 17$. Since the quota is 9, and 9 is more than 8.5 and less than 17, this system is valid.

d. $[16:5,4,3,3,1]$

The quota is 16 in this example. The total weight is $5+4+3+3+1=16$. Half of 16 is 8, so the quota must be $8 < q \leq 16$. Since the quota is 16, and 16 is equal to the maximum of the possible values of the quota, this system is valid. In this system, all of the players must vote in favor of a motion in order for the motion to pass.

e. $[9:10,3,2]$

The quota is 9 in this example. The total weight is $10+3+2=15$. Half of 15 is 7.5, so the quota must be $7.5 < q \leq 15$. Since the quota is 9, and 9 is between 7.5 and 15, this system is valid.

f. $[8:5,4,2]$

The quota is 8 in this example. The total weight is $5+4+2=11$. Half of 11 is 5.5, so the quota must be $5.5 < q \leq 11$. Since the quota is 8, and 8 is between 5.5 and 11, the system is valid.

In Example 7.2.2, some of the weighted voting systems are valid systems. Let's examine these for some concepts. In the system $[9:10,3,2]$, player one has a weight of 10. Since the quota is nine, this player can pass any motion it wants to. So, player one holds all the power. A player with all the power that can pass any motion alone is called a **dictator**. In the system $[16:5,4,3,3,1]$, every player has the same amount of power since all players are needed to pass a motion. That also means that any player can stop a motion from passing. A player that can stop a motion from passing is said to have **veto power**. In the system $[8:5,4,2]$, player three has a weight of two. Players one and two can join together and pass any motion without player three, and player three doesn't have enough weight to join with either player one or player two to pass a motion. So player three has no power. A player who has no power is called a **dummy**.

Example 7.2.3: Dictator, Veto Power, or Dummy?

In the weighted voting system $[57:23,21,16,12]$, are any of the players a dictator or a dummy or do any have veto power.

Since no player has a weight higher than or the same as the quota, then there is no dictator. If players one and two join together, they can't pass a motion without player three, so player three has veto power. Under the same logic, players one and two also have veto power. Player four cannot join with any players to pass a motion, so player four's votes do not matter. Thus, player four is a dummy.

Now that we have an understanding of some of the basic concepts, how do we quantify how much power each player has? There are two different methods. One is called the Banzhaf Power Index and the other is the Shapely-Shubik Power Index. We will look at each of these indices separately.

Banzhaf Power Index:

A coalition is a set of players that join forces to vote together. If there are three players P_1, P_2 and P_3 then the coalitions would be:

$\{P_1\},\{P_2\},\{P_3\},\{P_1,P_2\},\{P_1,P_3\},\{P_2,P_3\},\{P_1,P_2,P_3\}$.

Not all of these coalitions are winning coalitions. To find out if a coalition is winning or not look at the sum of the weights in each coalition and then compare that sum to the quota. If the sum is the quota or more, then the coalition is a winning coalition.

Example 7.2.4: Coalitions with Weights

In the weighted voting system $[17:12,7,3]$, the weight of each coalition and whether it wins or loses is in the table below.

Table 7.2.1: Coalition Listing

Coalition	Weight	Win or Lose?
$\{P_1\}$	12	Lose
$\{P_2\}$	7	Lose
$\{P_3\}$	3	Lose
$\{P_1, P_2\}$	19	Win
$\{P_1, P_3\}$	15	Lose
$\{P_2, P_3\}$	10	Lose
$\{P_1, P_2, P_3\}$	22	Win

In each of the winning coalitions you will notice that there may be a player or players that if they were to leave the coalition, the coalition would become a losing coalition. If there is such a player or players, they are known as the **critical player(s)** in that coalition.

Example 7.2.5: Critical Players

In the weighted voting system $[17:12,7,3]$, determine which player(s) are critical player(s). Note that we have already determined which coalitions are winning coalitions for this weighted voting system in Example 7.2.4. Thus, when we continue on to determine the critical player(s), we only need to list the winning coalitions.

Table 7.2.2: Winning Coalitions and Critical Players

Coalition	Weight	Win or Lose?	Critical Player
$\{P_1, P_2\}$	19	Win	P_1, P_2
$\{P_1, P_2, P_3\}$	22	Win	P_1, P_2

Notice, player one and player two are both critical players two times and player three is never a critical player.

Banzhaf Power Index:

The Banzhaf power index is one measure of the power of the players in a weighted voting system. In this index, a player's power is determined by the ratio of the number of times that player is critical to the total number of times any and all players are critical.

> **Banzhaf Power Index** for Player $P_i = \dfrac{B_i}{T}$
>
> where B_i = number of times player P_i is critical
>
> and T = total number of times all players are critical

Example 7.2.6: Banzhaf Power Index

In the weighted voting system $[17:12,7,3]$, determine the Banzhaf power index for each player.

Using table 7.2.2, Player one is critical two times, Player two is critical two times, and Player three is never critical. So $T = 4$, $B_1 = 2$, $B_2 = 2$, and $B_3 = 0$. Thus:

Banzhaf power index of P_1 is $\frac{2}{4} = \frac{1}{2} = 0.5 = 50\%$

Banzhaf power index of P_2 is $\frac{2}{4} = \frac{1}{2} = 0.5 = 50\%$

Banzhaf power index of P_3 is $\frac{0}{4} = 0 = 0\%$

So players one and two each have 50% of the power. This means that they have equal power, even though player one has five more votes than player two. Also, player three has 0% of the power and so player three is a dummy.

How many coalitions are there? From the last few examples, we know that if there are three players in a weighted voting system, then there are seven possible coalitions. How about when there are four players?

Table 7.2.3: Coalitions with Four Players

1 Player	**2 Players**	**3 Players**	**4 Players**
$\{P_1\},\{P_2\},\{P_3\},\{P_4\}$	$\{P_1,P_2\},\{P_1,P_3\},\{P_1,P_4\}$ $\{P_2,P_3\},\{P_2,P_4\},\{P_3,P_4\}$	$\{P_1,P_2,P_3\},\{P_1,P_2,P_4\}$ $\{P_1,P_3,P_4\},\{P_2,P_3,P_4\}$	$\{P_1,P_2,P_3,P_4\}$

So when there are four players, it turns out that there are 15 coalitions. When there are five players, there are 31 coalitions (there are too many to list, so take my word for it). It doesn't look like there is a pattern to the number of coalitions, until you realize that 7, 15, and 31 are all one less than a power of two. In fact, seven is one less than 2^3, 15 is one less than 2^4, and 31 is one less than 2^5. So it appears that the number of coalitions for N players is $2^N - 1$.

Example 7.2.7: Banzhaf Power Index

Example 7.2.1 had the weighted voting system of $[58:30,25,22,14,9]$. Find the Banzhaf power index for each player.

Since there are five players, there are 31 coalitions. This is too many to write out, but if we are careful, we can just write out the winning coalitions. No player can win alone, so we can ignore all of the coalitions with one player. Also, no two-player coalition can win either. So we can start with the three player coalitions.

Table 7.2.4: Winning Coalitions and Critical Players

Winning Coalition	Critical Player
$\{P_1,P_2,P_3\}$	P_1,P_2,P_3
$\{P_1,P_2,P_4\}$	P_1,P_2,P_4
$\{P_1,P_2,P_5\}$	P_1,P_2,P_5
$\{P_1,P_3,P_4\}$	P_1,P_3,P_4
$\{P_1,P_3,P_5\}$	P_1,P_3,P_5
$\{P_2,P_3,P_4\}$	P_2,P_3,P_4
$\{P_1,P_2,P_3,P_4\}$	
$\{P_1,P_2,P_3,P_5\}$	P_1
$\{P_1,P_2,P_4,P_5\}$	P_1,P_2
$\{P_1,P_3,P_4,P_5\}$	P_1,P_3
$\{P_2,P_3,P_4,P_5\}$	P_2,P_3,P_4
$\{P_1,P_2,P_3,P_4,P_5\}$	

So player one is critical eight times, player two is critical six times, player three is critical six times, player four is critical four times, and player five is critical two times. Thus, the total number of times any player is critical is $T = 26$.

Banzhaf power index for P_1 = $\frac{8}{26}=\frac{4}{13}$ = 0.308 = 30.8%

Banzhaf power index for P_2 = $\frac{6}{26}=\frac{3}{13}$ = 0.231 = 23.1%

Banzhaf power index for P_3 = $\frac{6}{26} = \frac{3}{13}$ = 0.231 = 23.1%

Banzhaf power index for P_4 = $\frac{4}{26} = \frac{2}{13}$ = 0.154 = 15.4%

Banzhaf power index for P_5 = $\frac{2}{26} = \frac{1}{13}$ = 0.077 = 7.7%

Every player has some power. Player one has the most power with 30.8% of the power. No one has veto power, since no player is in every winning coalition.

Shapley-Shubik Power Index:

Shapley-Shubik takes a different approach to calculating the power. Instead of just looking at which players can form coalitions, Shapley-Shubik decided that all players form a coalition together, but the order that players join a coalition is important. This is called a **sequential coalition**. Instead of looking at a player leaving a coalition, this method examines what happens when a player joins a coalition. If when a player joins the coalition, the coalition changes from a losing to a winning coalition, then that player is known as a **pivotal player**. Now we count up how many times each player is pivotal, and then divide by the number of sequential coalitions. Note, that in reality when coalitions are formed for passing a motion, not all players will join the coalition. The sequential coalition is used only to figure out the power each player possesses.

As an example, suppose you have the weighted voting system of $[17:12,7,3]$. One of the sequential coalitions is $\langle P_1, P_2, P_3 \rangle$ which means that P_1 joins the coalition first, followed by P_2 joining the coalition, and finally, P_3 joins the coalition. When player one joins the coalition, the coalition is a losing coalition with only 12 votes. Then, when player two joins, the coalition now has enough votes to win (12 + 7 = 19 votes). Player three joining doesn't change the coalition's winning status so it is irrelevant. Thus, player two is the pivotal player for this coalition. Another sequential coalition is $\langle P_1, P_3, P_2 \rangle$. When player one joins the coalition, the coalition is a losing coalition with only 12 votes. Then player three joins but the coalition is still a losing coalition with only 15 votes. Then player two joins and the coalition is now a winning coalition with 22 votes. So, player two is the pivotal player for this coalition as well.

How many sequential coalitions are there for N players? Let's look at three players first. The sequential coalitions for three players (P_1, P_2, P_3) are:
$\langle P_1, P_2, P_3 \rangle, \langle P_1, P_3, P_2 \rangle, \langle P_2, P_1, P_3 \rangle, \langle P_2, P_3, P_1 \rangle, \langle P_3, P_1, P_2 \rangle, \langle P_3, P_2, P_1 \rangle$.

Note: The difference in notation: We use $\{\ \}$ *for coalitions and* $\langle\ \rangle$ *sequential coalitions.*

So there are six sequential coalitions for three players. Can we come up with a mathematical formula for the number of sequential coalitions? For the first player in the sequential coalition, there are 3 players to choose from. Once you choose one for the first spot, then there are only 2 players to choose from for the second spot. The third spot will only have one player to put in that spot. Notice, 3*2*1 = 6. It looks like if you have N players, then you can find the number of sequential coalitions by multiplying $N(N-1)(N-2)\cdots(3)(2)(1)$. This expression is called a *N* factorial, and is denoted by *N!*.

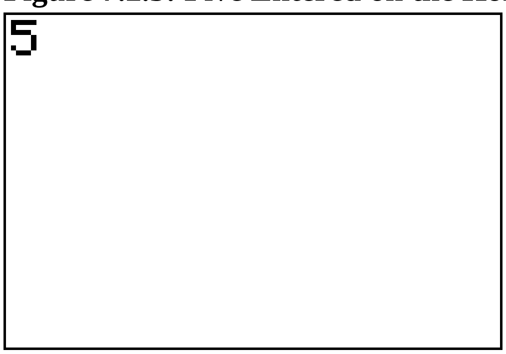
Most calculators have a factorial button. The process for finding a factorial on the TI-83/84 is demonstrated in the following example.

Example 7.2.8: Finding a Factorial on the TI-83/84 Calculator

Find 5! on the TI-83/84 Calculator.

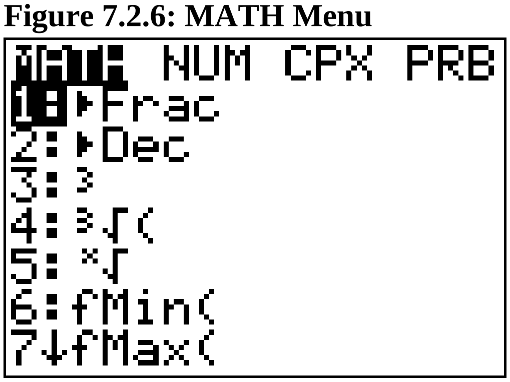
First, note that $5! = 5*4*3*2*1$, which is easy to do without the special button on the calculator, be we will use it anyway. First, input the number five on the home screen of the calculator.

Figure 7.2.5: Five Entered on the Home Screen



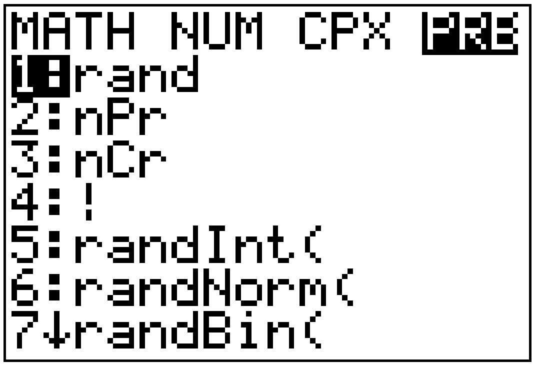
Then press the MATH button. You will see the following:

Figure 7.2.6: MATH Menu



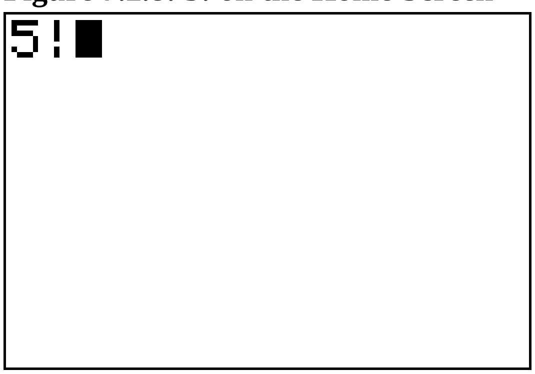
Now press the right arrow key to move over to the abbreviation PRB, which stands for probability.

Figure 7.2.7: PRB Menu



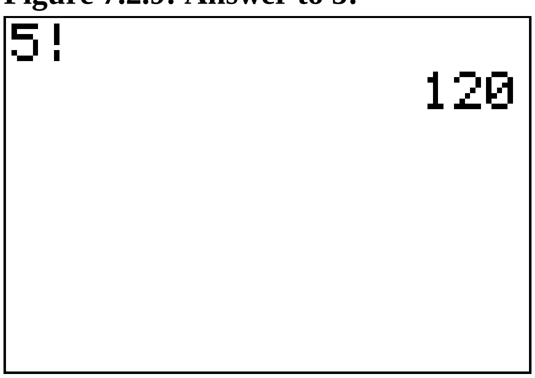
Number 4:! is the factorial button. Either arrow down to the number four and press ENTER, or just press the four button. This will put the ! next to your five on the home screen.

Figure 7.2.8: 5! on the Home Screen



Now press ENTER and you will see the result.

Figure 7.2.9: Answer to 5!



Notice that 5! is a very large number. So if you have 5 players in the weighted voting system, you will need to list 120 sequential coalitions. This is quite large, so most calculations using the Shapley-Shubik power index are done with a computer.

Now we have the concepts for calculating the Shapley-Shubik power index.

Shapley-Shubik Power Index for Player $P_i = \frac{S_i}{N!}$

where S_i is how often the player is pivotal

N is the number of players and *N!* is the number of sequential coalitions

Example 7.2.9: Shapley-Shubik Power Index

In the weighted voting system $[17:12,7,3]$, determine the Shapley-Shubik power index for each player.

First list every sequential coalition. Then determine which player is pivotal in each sequential coalition. There are 3! = 6 sequential coalitions.

Table 7.2.10: Sequential Coalitions and Pivotal Players

Sequential coalition	Pivotal player
$\langle P_1, P_2, P_3 \rangle$	P_2
$\langle P_1, P_3, P_2 \rangle$	P_2
$\langle P_2, P_1, P_3 \rangle$	P_1
$\langle P_2, P_3, P_1 \rangle$	P_1
$\langle P_3, P_1, P_2 \rangle$	P_2
$\langle P_3, P_2, P_1 \rangle$	P_1

So, $S_1 = 3$, $S_2 = 3$, and $S_3 = 0$.

Shapley-Shubik power index for $P_1 = \frac{3}{6} = \frac{1}{2}$ = 0.5 = 50%

Shapley-Shubik power index for $P_2 = \frac{3}{6} = \frac{1}{2}$ = 0.5 = 50%

Shapley-Shubik power index for $P_3 = \frac{0}{6} = 0$ = 0%

This is the same answer as the Banzhaf power index. The two methods will not usually produce the same exact answer, but their answers will be close to the same value. Notice that player three is a dummy using both indices.

Example 7.2.10: Calculating the Power

For the voting system $[7:6,4,2]$, find:

a. The Banzhaf power index for each player

The first thing to do is list all of the coalitions and determine which ones are winning and which ones are losing. Then determine the critical player(s) in each winning coalition.

Table 7.2.11 Coalitions and Critical Players

Coalition	Weight	Win or Lose?	Critical Player
$\{P_1\}$	6	Lose	
$\{P_2\}$	4	Lose	
$\{P_3\}$	2	Lose	
$\{P_1, P_2\}$	10	Win	P_1, P_2
$\{P_1, P_3\}$	8	Win	P_1, P_3
$\{P_2, P_3\}$	6	Lose	
$\{P_1, P_2, P_3\}$	12	Win	P_1

So, $B_1 = 3,\ B_2 = 1,\ B_3 = 1,\ T = 3+1+1 = 5$

Banzhaf power index of $P_1 = \frac{3}{5}$ = 0.6 = 60%

Banzhaf power index of $P_2 = \frac{1}{5}$ = 0.2 = 20%

Banzhaf power index of $P_3 = \frac{1}{5}$ = 0.2 = 20%

b. The Shapley-Shubik power index for each player

The first thing to do is list all of the sequential coalitions, and then determine the pivotal player in each sequential coalition.

Table 7.2.12: Sequential Coalitions and Pivotal Players

Sequential Coalition	**Pivotal Player**
$\langle P_1, P_2, P_3 \rangle$	*P2*
$\langle P_1, P_3, P_2 \rangle$	*P3*
$\langle P_2, P_1, P_3 \rangle$	*P1*
$\langle P_2, P_3, P_1 \rangle$	*P1*
$\langle P_3, P_1, P_2 \rangle$	*P1*
$\langle P_3, P_2, P_1 \rangle$	*P1*

So $S_1 = 4,\ S_2 = 1,\ S_3 = 1,\ 3! = 6$

Shapley-Shubik power index of *P1* $= \frac{4}{6} = \frac{2}{3}$ = 0.667 = 66.7%

Shapley-Shubik power index of *P2* $= \frac{1}{6}$ = 0.167 = 16.7%

Shapley-Shubik power index of *P3* $= \frac{1}{6}$ = 0.167 = 16.7%

Notice the two indices give slightly different results for the power distribution, but they are close to the same values.

Chapter 7 Homework

1. An organization recently made a decision about which company to use to redesign its website and host its members' information. The Board of Directors will vote using preference ballots ranking their first choice to last choice of the following companies: Allied Web Design (A), Ingenuity Incorporated (I), and Yeehaw Web Trends (Y). The individual ballots are shown below. Create a preference schedule summarizing these results.
   AIY, YIA, YAI, AIY, YIA, IAY, IYA, IAY, YAI, YIA, AYI, YIA, YAI

2. A group needs to decide where their next conference will be held. The choices are Kansas City (K), Lafayette (L), and Minneapolis (M). The individual ballots are shown below. Create a preference schedule summarizing these results.
   KLM, LMK, MLK, LMK, MKL, KLM, KML, LMK, MKL, MKL, MLK, MLK

3. A book club holds a vote to figure out what book they should read next. They are picking from three different books. The books are labeled A, B, and C, and the preference schedule for the vote is below.

Number of voters	12	9	8	5	10
1st choice	A	B	B	C	C
2nd choice	C	A	C	A	B
3rd choice	B	C	A	B	A

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

4. An election is held for a new vice president at a college. There are three candidates (A, B, C), and the faculty rank which candidate they like the most. The preference ballot is below.

Number of voters	8	10	12	9	4	1
1st choice	A	A	B	B	C	C
2nd choice	B	C	A	C	A	B
3rd choice	C	B	C	A	B	A

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

5. A city election for a city council seat was held between 4 candidates, Martorana (M), Jervey (J), Riddell (R), and Hanrahan (H). The preference schedule for this election is below.

Number of voters	60	73	84	25	110
1st choice	M	M	H	J	J
2nd choice	R	H	R	R	M
3rd choice	H	R	M	M	R
4th choice	J	J	J	H	H

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

6. A local advocacy group asks members of the community to vote on which project they want the group to put its efforts behind. The projects are green spaces (G), city energy code (E), water conservation (W), and promoting local business (P). The preference schedule for this vote is below.

Number of voters	12	57	23	34	13	18	22	39
1st choice	W	W	G	G	E	E	P	P
2nd choice	G	P	W	E	G	P	W	E
3rd choice	E	E	E	W	P	W	G	W
4th choice	P	G	P	P	W	G	E	G

   a. How many voters voted in the election?
   b. How many votes are needed for a majority?
   c. Find the winner using the Plurality Method.
   d. Find the winner using the Borda Count Method.
   e. Find the winner using the Plurality with Elimination Method.
   f. Find the winner using the Pairwise Comparisons Method.

7. An election is held and candidate A wins. A mistake was found that showed that candidate C was not qualified to run in the election. The candidate was removed, and the election office determined the winner with candidate C removed. Now candidate D wins. What fairness criterion was violated?

8. An election is held and candidate C wins. Before the election is certified the ballots are misplaced. Another election is held, and the only change in the ballots was that more people put C as their first choice. When the winner is determined, candidate A now wins. What fairness criterion was violated?

9. An election is held and candidate B has the majority of first-place votes. However, candidate B does not win the election. What fairness criterion was violated?

10. An election is held and candidate D is favored in a head-to-head comparison to every other candidate. However, D does not win the election. What is D called, and what fairness criterion was violated?

11. Consider the weighted voting system $[15:7,6,3,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

12. Consider the weighted voting system $[23:10,3,2,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

13. Consider the weighted voting system $[9:10,3,2,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

14. Consider the weighted voting system $[16:9,6,3,1]$.
    a. How many players are there?
    b. List the weight of each player.
    c. What is the quota?
    d. Is this a valid system? Why or why not?

15. Consider the weighted voting system $[q:7,6,4,1]$.
    a. What is the minimum value of the quota q?
    b. What is the maximum value of the quota q?
    c. What is the quota q if a motion can only pass with 2/3's of the vote?
    d. What is the quota q if a motion can only pass with more than 2/3's of the vote?

16. Consider the weighted voting system $[q: 25, 20, 15, 15, 6]$.
    a. What is the minimum value of the quota q?
    b. What is the maximum value of the quota q?
    c. What is the quota q if a motion can only pass with 2/3's of the vote?
    d. What is the quota q if a motion can only pass with more than 2/3's of the vote?

17. Consider the weighted voting system $[12: 13, 5, 4, 1]$. Are any players dictators? Explain.

18. Consider the weighted voting system $[16: 12, 2, 2, 1]$. Do any players have veto power? Explain.

19. Consider the weighted voting system $[24: 19, 16, 12]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapley-Shubik power index for each player.
    c. Are any players a dummy?

20. Consider the weighted voting system $[54: 42, 13, 12]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapley-Shubik power index for each player.
    c. Are any players a dummy?

21. Consider the weighted voting system $[13: 7, 6, 2]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapley-Shubik power index for each player.
    c. Are any players a dummy?

22. Consider the weighted voting system $[15: 16, 12, 1]$.
    a. Find the Banzhaf power index for each player.
    b. Find the Shapley-Shubik power index for each player.

c. Are any players a dummy?

23. Consider the weighted voting system $[16:12,2,2,1]$. Find the Banzhaf power index for each player?

24. Consider the weighted voting system $[16:9,6,3,1]$. Find the Banzhaf power index for each player?

25. The United Nations (UN) Security Council consists of five permanent members (United States, Russian Federation, the United Kingdom, France, and China) and 10 non-permanent members elected for two-year terms by the General Assembly. The five permanent members have veto power, and a resolution cannot pass without nine members voting for it. Set up the weighted voting system for the UN.