

### Chapter 4: Growth

Population growth is a current topic in the media today. The world population is growing by over 70 million people every year. Predicting populations in the future can have an impact on how countries plan to manage resources for more people. The tools needed to help make predictions about future populations are growth models like the exponential function. This chapter will discuss real world phenomena, like population growth and radioactive decay, using three different growth models.

The growth functions to be examined are linear, exponential, and logistic growth models. Each type of model will be used when data behaves in a specific way and for different types of scenarios. Data that grows by the same amount in each iteration uses a different model than data that increases by a percentage.

#### Section 4.1: Linear Growth

Starting at the age of 25, imagine if you could save \$20 per week, every week, until you retire, how much money would you have stuffed under your mattress at age 65? To solve this problem, we could use a linear growth model. Linear growth has the characteristic of growing by the same amount in each unit of time. In this example, there is an increase of \$20 per week; a constant amount is placed under the mattress in the same unit of time.

If we start with \$0 under the mattress, then at the end of the first year we would have  $\$20 \times 52 = \$1040$ . So, this means you could add \$1040 under your mattress every year. At the end of 40 years, you would have  $\$1040 \times 40 = \$41,600$  for retirement. This is not the best way to save money, but we can see that it is calculated in a systematic way.

**Linear Growth:** A quantity grows linearly if it grows by a constant amount for each unit of time.

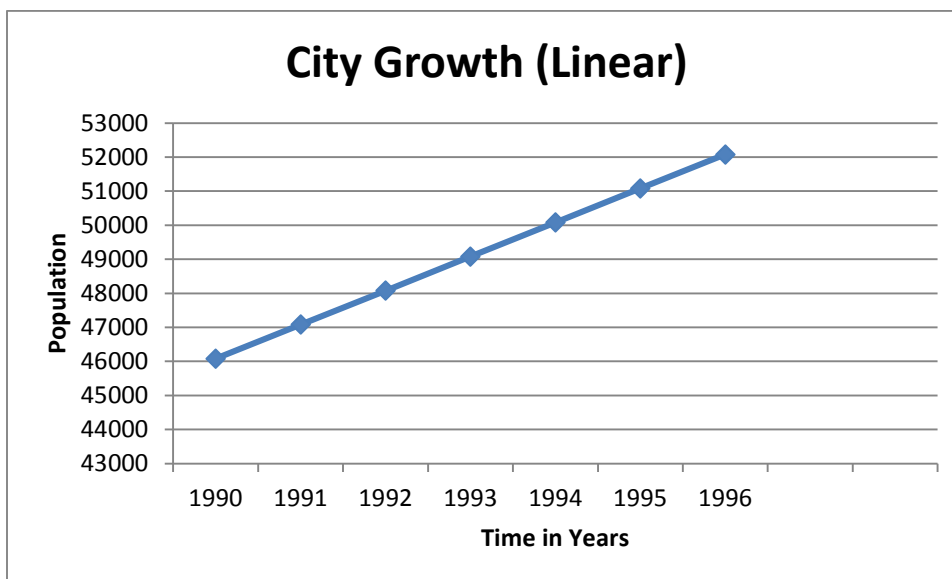
#### Example 4.1.1: City Growth

Suppose in Flagstaff Arizona, the number of residents increased by 1000 people per year. If the initial population was 46,080 in 1990, can you predict the population in 2013? This is an example of linear growth because the population grows by a constant amount. We list the population in future years below by adding 1000 people for each passing year.

	1990	1991	1992	1993	1994	1995	1996
Year	0	1	2	3	4	5	6
Population	46,080	47,080	48,080	49,080	50,080	51,080	52,080

**Figure 4.1.1: Graph of Linear Population Growth**

This is the graph of the population growth over a six year period in Flagstaff, Arizona. It is a straight line and can be modeled with a linear growth model.



The population growth can be modeled with a linear equation. The initial population  $P_0$  is 46,080. The future population depends on the number of years,  $t$ , after the initial year. The model is  $P(t) = 46,080 + 1000t$

To predict the population in 2013, we identify how many years it has been from 1990, which is year zero. So  $n = 23$  for the year 2013.

$$P(23) = 46,080 + 1000(23) = 69,080$$

The population of Flagstaff in 2013 would be 69,080 people.

**Linear Growth Model:** Linear growth begins with an initial population called  $P_0$ . In each time period or generation  $t$ , the population changes by a constant amount called the common difference  $d$ . The basic model is:

$$P(t) = P_0 + td$$

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### Example 4.1.2: Antique Frog Collection

Dora has inherited a collection of 30 antique frogs. Each year she vows to buy two frogs a month to grow the collection. This is an additional 24 frogs per year. How many frogs will she have in six years? How long will it take her to reach 510 frogs?

The initial population is  $P_0 = 30$  and the common difference is  $d = 24$ . The linear growth model for this problem is:

$$P(t) = 30 + 24t$$

The first question asks how many frogs Dora will have in six years so,  $t = 6$ .

$$P(6) = 30 + 24(6) = 30 + 144 = 174 \text{ frogs.}$$

The second question asks for the time it will take for Dora to collect 510 frogs.

So,  $P(t) = 510$  and we will solve for  $t$ .

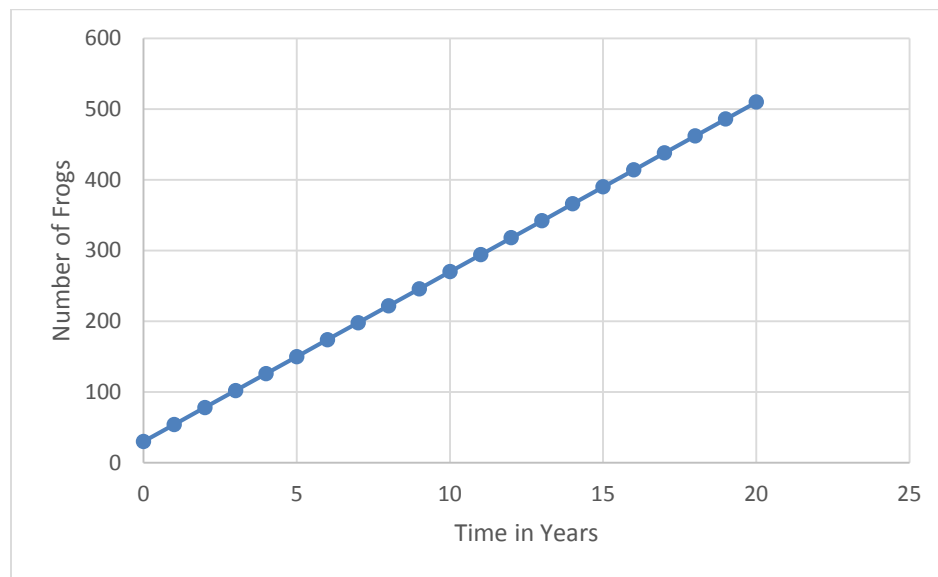
$$510 = 30 + 24t$$

$$480 = 24t$$

$$20 = t$$

It will take 20 years to collect 510 antique frogs.

**Figure 4.1.2: Graph of Antique Frog Collection**



*Note: The graph of the number of antique frogs Dora accumulates over time follows a straight line.*

**Example 4.1.3: Car Depreciation**

Assume a car depreciates by the same amount each year. Joe purchased a car in 2010 for \$16,800. In 2014 it is worth \$12,000. Find the linear growth model. Predict how much the car will be worth in 2020.

$$P_0 = 16,800 \text{ and } P(4) = 12,000$$

To find the linear growth model for this problem, we need to find the common difference  $d$ .

$$P(t) = P_0 + td$$

$$12,000 = 16,800 + 4d$$

$$-4800 = 4d$$

$$-1200 = d$$

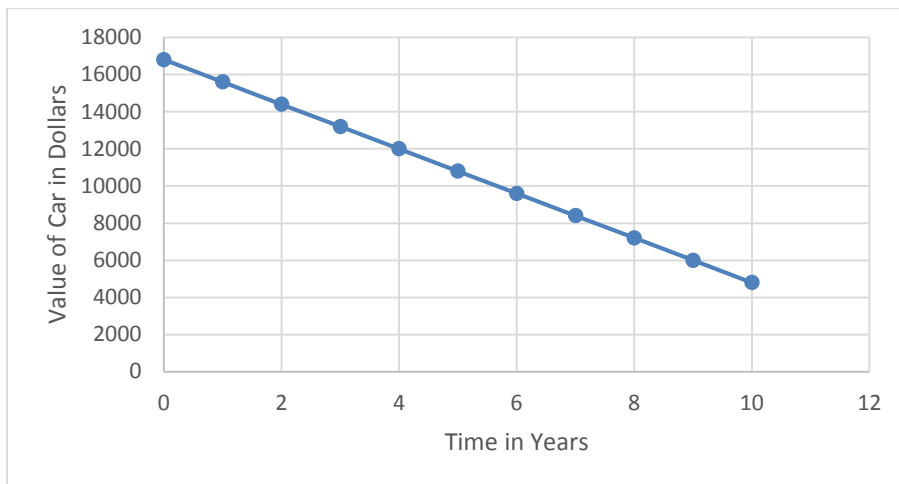
The common difference of depreciation each year is  $d = \$-1200$ . Thus, the linear growth model for this problem is:  $P(t) = 16,800 - 1200t$

Now, to find out how much the car will be worth in 2020, we need to know how many years that is from the purchase year. Since it is ten years later,  $t = 10$ .

$$P(10) = 16,800 - 1200(10) = 16,800 - 12,000 = 4,800$$

The car is worth \$4800 in 2020.

**Figure 4.1.3: Graph of Car Value Depreciation**



*Note: The value of the car over time follows a decreasing straight line.*

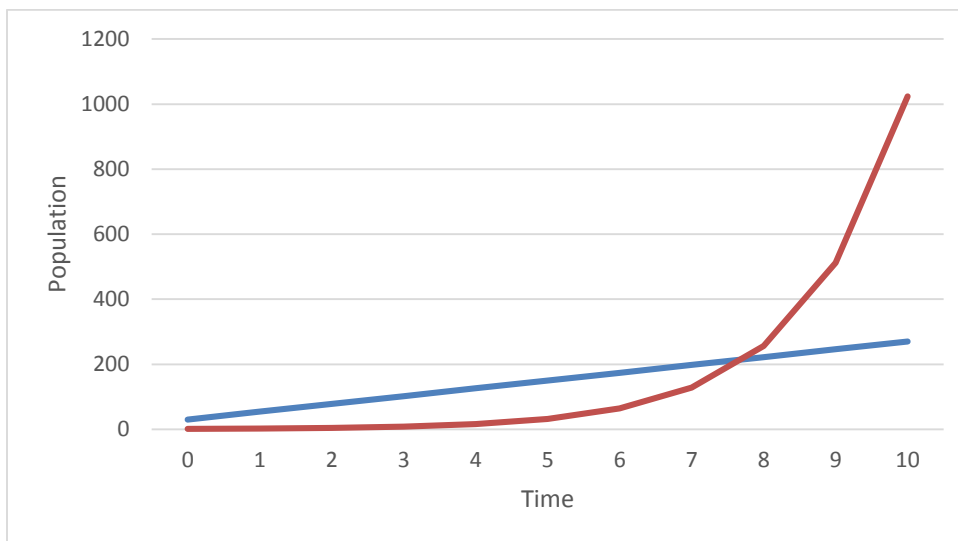
### Section 4.2: Exponential Growth

The next growth we will examine is exponential growth. Linear growth occurs by adding the same amount in each unit of time. Exponential growth happens when an initial population increases by the same percentage or factor over equal time increments or generations. This is known as relative growth and is usually expressed as percentage. For example, let's say a population is growing by 1.6% each year. For every 1000 people in the population, there will be  $1000 \times 0.016 = 16$  more people added per year.

**Exponential Growth:** A quantity grows exponentially if it grows by a constant factor or rate for each unit of time.

#### Figure 4.2.1: Graphical Comparison of Linear and Exponential Growth

In this graph, the blue straight line represents linear growth and the red curved line represents exponential growth.



#### Example 4.2.1: City Growth

A city is growing at a rate of 1.6% per year. The initial population in 2010 is  $P_0 = 125,000$ . Calculate the city's population over the next few years.

The relative growth rate is 1.6%. This means an additional 1.6% is added on to 100% of the population that already exists each year. This is a factor of 101.6%.

$$\text{Population in 2011} = 125,000(1.016)^1 = 127,000$$

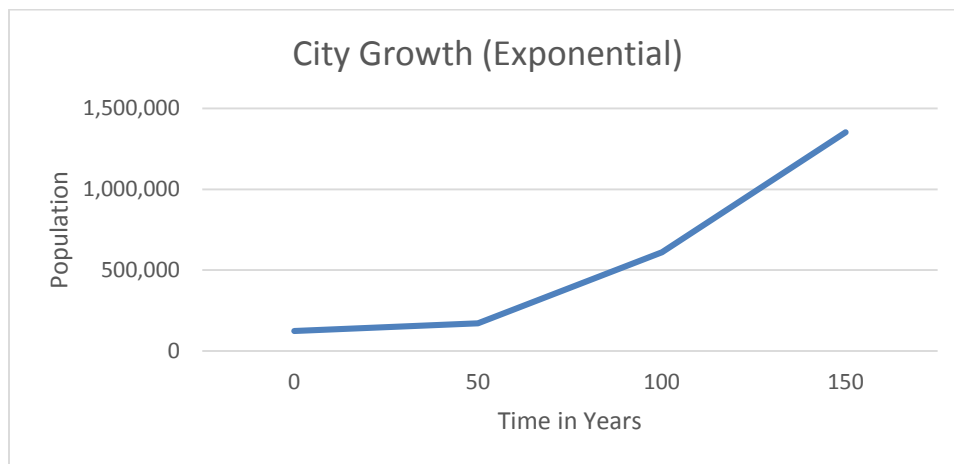
$$\text{Population in 2012} = 127,000(1.016) = 125,000(1.016)^2 = 129,032$$

$$\text{Population in 2013} = 129,032(1.016) = 125,000(1.016)^3 = 131,097$$

We can create an equation for the city's growth. Each year the population is 101.6% more than the previous year.

$$P(t) = 125,000(1 + 0.016)^t$$

**Figure 4.2.2: Graph of City Growth**



*Note: The graph of the city growth follows an exponential growth model.*

#### Example 4.2.2: A Shrinking Population

St. Louis, Missouri has declined in population at a rate of 1.6 % per year over the last 60 years. The population in 1950 was 857,000. Find the population in 2014. (Wikipedia, n.d.)

$$P_0 = 857,000$$

The relative growth rate is 1.6%. This means 1.6% of the population is subtracted from 100% of the population that already exists each year. This is a factor of 98.4%.

$$\text{Population in 1951} = 857,000(0.984)^1 = 843,288$$

$$\text{Population in 1952} = 843,228(0.984) = 857,000(0.984)^2 = 829,795$$

$$\text{Population in 1953} = 828,795(0.984) = 857,000(0.984)^3 = 816,519$$

We can create an equation for the city's growth. Each year the population is 1.6% less than the previous year.  $P(t) = 857,000(1 - 0.016)^t$

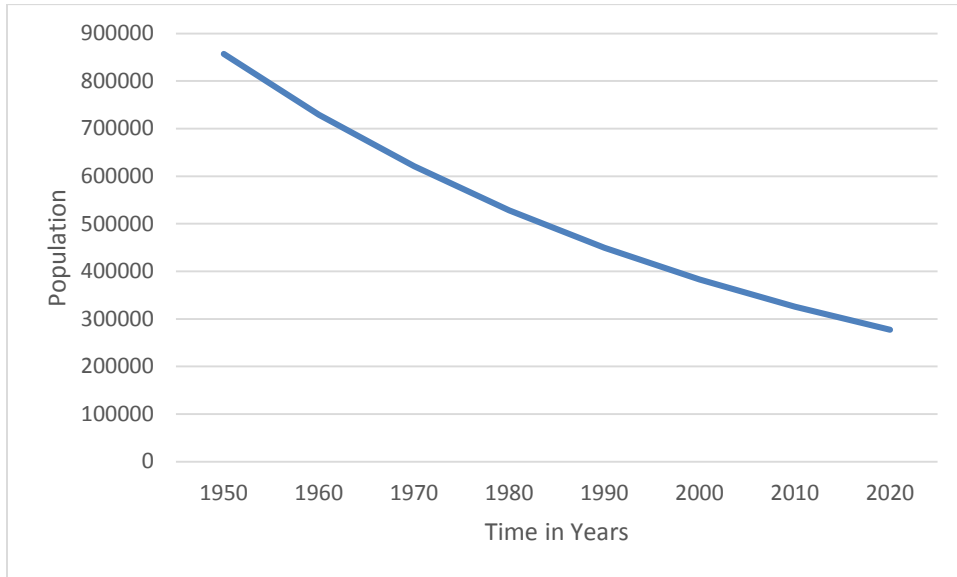
So the population of St. Louis Missouri in 2014, when  $t = 64$ , is:

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$$\begin{aligned}P(64) &= 857,000(1 - 0.016)^{64} \\ &= 857,000(0.984)^{64} \\ &= 305,258\end{aligned}$$

**Figure 4.2.3: Graph of St. Louis, Missouri Population Decline**



*Note: The graph of the population of St. Louis, Missouri over time follows a declining exponential growth model.*

**Exponential Growth Model**  $P(t) = P_0(1 + r)^t$

$P_0$  is the initial population,

$r$  is the relative growth rate.

$t$  is the time unit.

$r$  is positive if the population is increasing and negative if the population is decreasing

### Example 4.2.3: Inflation

The average inflation rate of the U.S. dollar over the last five years is 1.7% per year. If a new car cost \$18,000 five years ago, how much would it cost today? (U.S. Inflation Calculator, n.d.)

To solve this problem, we use the exponential growth model with  $r = 1.7\%$ .

$$P_0 = 18,000 \text{ and } t = 5$$

$$P(t) = 18,000(1 + 0.017)^t$$

$$P(5) = 18,000(1 + 0.017)^5 = 19,582.91$$

This car would cost \$19,582.91 today.

#### Example 4.2.4: Ebola Epidemic in Sierra Leone

In May of 2014 there were 15 cases of Ebola in Sierra Leone. By August, there were 850 cases. If the virus is spreading at the same rate (exponential growth), how many cases will there be in February of 2015? (McKenna, 2014)

To solve this problem, we have to find three things; the growth rate per month, the exponential growth model, and the number of cases of Ebola in February 2015.

First calculate the growth rate per month. To do this, use the initial population

$P_0 = 15$ , in May 2014. Also, in August, three months later, the number of cases was 850 so,  $P(3) = 850$ .

Use these values and the exponential growth model to solve for  $r$ .

$$P(t) = P_0(1 + r)^t$$

$$850 = 15(1 + r)^3$$

$$56.67 = (1 + r)^3$$

$$\sqrt[3]{56.67} = \sqrt[3]{(1 + r)^3}$$

$$3.84 = 1 + r$$

$$2.84 = r$$

The growth rate is 284% per month. Thus, the exponential growth model is:

$$P(t) = 15(1 + 2.84)^t = 15(3.84)^t$$

Now, we use this to calculate the number of cases of Ebola in Sierra Leone in February 2015, which is 9 months after the initial outbreak so,  $t = 9$ .

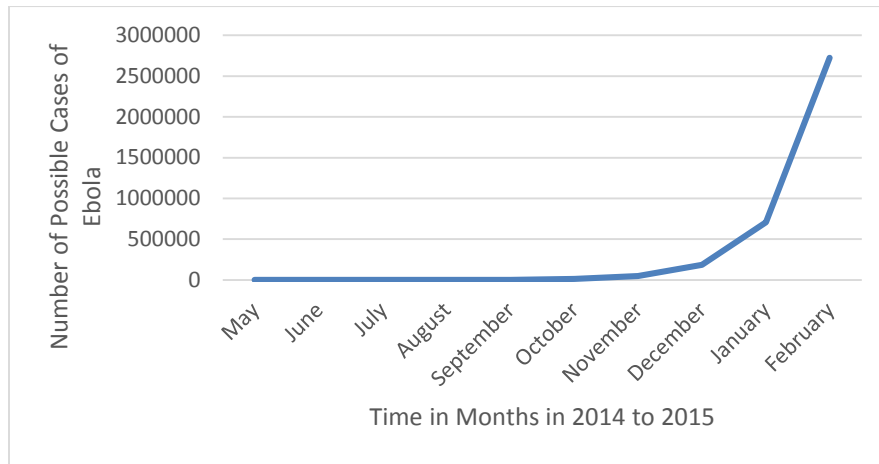
$$P(9) = 15(3.84)^9 = 2,725,250$$

If this same exponential growth rate continues, the number of Ebola cases in Sierra Leone in February 2015 would be 2,725,250.

This is a bleak prediction for the community of Sierra Leone. Fortunately, the growth rate of this deadly virus should be reduced by the world community and World Health Organization by providing the needed means to fight the initial spread.



**Figure 4.2.4: Graph of Ebola Virus in Sierra Leone**



*Note: The graph of the number of possible Ebola cases in Sierra Leone over time follows an increasing exponential growth model.*

### **Example 4.2.5: Population Decline in Puerto Rico**

According to a new forecast, the population of Puerto Rico is in decline. If the population in 2010 is 3,978,000 and the prediction for the population in 2050 is 3,697,000, find the annual percent decrease rate. (Bloomberg Businessweek, n.d.) To solve this problem we use the exponential growth model. We need to solve for  $r$ .

$$P(t) = P_0(1+r)^t \text{ where } t = 40 \text{ years}$$

$$P(40) = 3,697,000 \text{ and } P_0 = 3,978,000$$

$$3,697,000 = 3,978,000(1+r)^{40}$$

$$0.92936 = (1+r)^{40}$$

$$\sqrt[40]{0.92936} = \sqrt[40]{(1+r)^{40}}$$

$$0.99817 = 1+r$$

$$-0.0018 = r$$

The annual percent decrease is 0.18%.

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### Section 4.3: Special Cases: Doubling Time and Half-Life

**Example 4.3.1: April Fool's Joke**

Let's say that on April 1<sup>st</sup> I say I will give you a penny, on April 2<sup>nd</sup> two pennies, four pennies on April 3<sup>rd</sup>, and that I will double the amount each day until the end of the month. How much money would I have agreed to give you on April 30?

With  $P_0 = \$0.01$ , we get the following table:

**Table 4.3.1: April Fool's Joke**

Day	Dollar Amount
April 1 = $P_0$	0.01
April 2 = $P_1$	0.02
April 3 = $P_2$	0.04
April 4 = $P_3$	0.08
April 5 = $P_4$	0.16
April 6 = $P_5$	0.32
....	....
April 30 = $P_{29}$	?

In this example, the money received each day is 100% more than the previous day. If we use the exponential growth model  $P(t) = P_0(1+r)^t$  with  $r = 1$ , we get the doubling time model.

$$P(t) = P_0(1+1)^t = P_0(2)^t$$

We use it to find the dollar amount when  $t = 29$  which represents April 30

$$P(29) = 0.01(2)^{29} = \$5,368,709.12$$

Surprised?

That is a lot of pennies.

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### Doubling Time Model:

#### Example 4.3.2: E. coli Bacteria

A water tank up on the San Francisco Peaks is contaminated with a colony of 80,000 E. coli bacteria. The population doubles every five days. We want to find a model for the population of bacteria present after  $t$  days. The amount of time it takes the population to double is five days, so this is our time unit. After  $t$  days have passed, then  $\frac{t}{5}$  is the number of time units that have passed. Starting with the initial amount of 80,000 bacteria, our doubling model becomes:

$$P(t) = 80,000(2)^{\frac{t}{5}}$$

Using this model, how large is the colony in two weeks' time? We have to be careful that the units on the times are the same; 2 weeks = 14 days.

$$P(14) = 80,000(2)^{\frac{14}{5}} = 557,152$$

The colony is now 557,152 bacteria.

**Doubling Time Model:** If  $D$  is the doubling time of a quantity (the amount of time it takes the quantity to double) and  $P_0$  is the initial amount of the quantity then the amount of the quantity present after  $t$  units of time is  $P(t) = P_0(2)^{\frac{t}{D}}$

#### Example 4.3.3: Flies

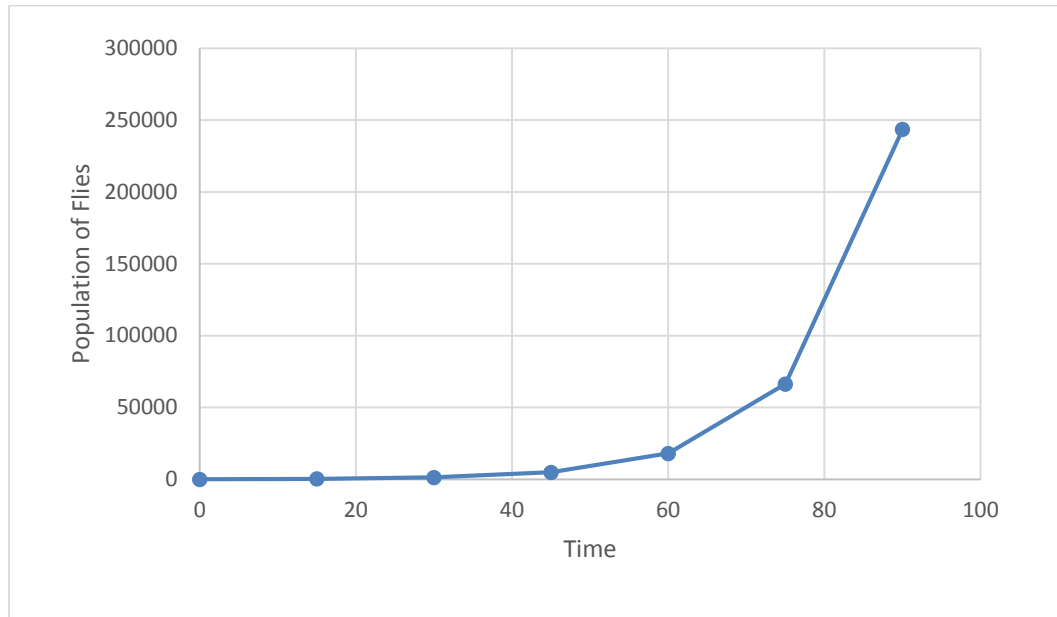
The doubling time of a population of flies is eight days. If there are initially 100 flies, how many flies will there be in 17 days? To solve this problem, use the doubling time model with  $D = 8$  and  $P_0 = 100$  so the doubling time model for this problem is:

$$P(t) = 100(2)^{\frac{t}{8}}$$

When  $t = 17$  days,

$$P(17) = 100(2)^{\frac{17}{8}} = 436$$

There are 436 flies after 17 days.

**Figure 4.3.2: Graph of Fly Population in Three Months**

*Note: The population of flies follows an exponential growth model.*

Sometimes we want to solve for the length of time it takes for a certain population to grow given their doubling time. To solve for the exponent, we use the *log* button on the calculator.

#### **Example 4.3.4: Bacteria Growth**

Suppose that a bacteria population doubles every six hours. If the initial population is 4000 individuals, how many hours would it take the population to increase to 25,000?

$P_0 = 4000$  and  $D = 6$ , so the doubling time model for this problem is:

$$P(t) = 4000(2)^{\frac{t}{6}} \quad \text{Now, find } t \text{ when } P(t) = 25,000$$

$$25,000 = 4000(2)^{\frac{t}{6}}$$

$$\frac{25,000}{4000} = \frac{4000(2)^{\frac{t}{6}}}{4000}$$

$$6.25 = (2)^{\frac{t}{6}} \quad \text{Now, take the log of both sides of the equation.}$$

$$\log 6.25 = \log (2)^{\frac{t}{6}} \quad \text{The exponent comes down using rules of logarithms.}$$

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$\log 6.25 = \left(\frac{t}{6}\right) \log 2$  Now, calculate  $\log 6.25$  and  $\log 2$  with your calculator.

$$0.7959 = \left(\frac{t}{6}\right) \times 0.3010$$

$$\frac{0.7959}{0.3010} = \frac{t}{6}$$

$$2.644 = \frac{t}{6}$$

$$t = 15.9$$

The population would increase to 25,000 bacteria in approximately 15.9 hours.

### Rule of 70:

There is a simple formula for approximating the doubling time of a population. It is called the rule of 70 and it is an approximation for growth rates less than 15%. Do not use this formula if the growth rate is 15% or greater.

**Rule of 70:** For a quantity growing at a constant percentage rate (not written as a decimal),  $R$ , per time period, the doubling time is approximately given by

$$\text{Doubling time } D \approx \frac{70}{R}$$

### Example 4.3.5: Bird Population

A bird population on a certain island has an annual growth rate of 2.5% per year. Approximate the number of years it will take the population to double. If the initial population is 20 birds, use it to find the bird population of the island in 17 years.

To solve this problem, first approximate the population doubling time.

$$\text{Doubling time } D \approx \frac{70}{2.5} = 28 \text{ years.}$$

With the bird population doubling in 28 years, we use the doubling time model to find the population is 17 years.

$$P(t) = 20(2)^{\frac{t}{28}}$$

When  $t = 17$  years

$$P(17) = 20(2)^{\frac{17}{28}} = 30.46$$

There will be 30 birds on the island in 17 years.

**Example 4.3.6: Cancer Growth Rate**

A certain cancerous tumor doubles in size every six months. If the initial size of the tumor is four cells, how many cells will there in three years? In seven years?

To calculate the number of cells in the tumor, we use the doubling time model. Change the time units to be the same. The doubling time is six months = 0.5 years.

$$P(t) = 4(2)^{\frac{t}{0.5}}$$

When  $t = 3$  years

$$P(3) = 4(2)^{\frac{3}{0.5}} = 256 \text{ cells}$$

When  $t = 7$  years

$$P(7) = 4(2)^{\frac{7}{0.5}} = 65,536 \text{ cells}$$

**Example 4.3.7: Approximating Annual Growth Rate**

Suppose that a certain city's population doubles every 12 years. What is the approximate annual growth rate of the city?

By solving the doubling time model for the growth rate, we can solve this problem.

$$D \approx \frac{70}{R}$$

$$R \cdot D \approx \frac{70}{\cancel{R}} \cdot \cancel{R}$$

$$RD \approx 70$$

$$\frac{R\cancel{D}}{\cancel{D}} \approx \frac{70}{D}$$

$$\text{Annual growth rate } R \approx \frac{70}{D}$$

$$R = \frac{70}{12} = 5.83\%$$

The annual growth rate of the city is approximately 5.83%

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### Exponential Decay and Half-Life Model:

The half-life of a material is the time it takes for a quantity of material to be cut in half. This term is commonly used when describing radioactive metals like uranium or plutonium. For example, the half-life of carbon-14 is 5730 years.

If a substance has a half-life, this means that half of the substance will be gone in a unit of time. In other words, the amount decreases by 50% per unit of time. Using the exponential growth model with a decrease of 50%, we have

$$P(t) = P_0(1-0.5)^t = P_0(0.5)^t = P_0\left(\frac{1}{2}\right)^t$$

#### Example 4.3.8: Half-Life

Let's say a substance has a half-life of eight days. If there are 40 grams present now, how much is left after three days? We want to find a model for the quantity of the substance that remains after  $t$  days. The amount of time it takes the quantity to be reduced by half is eight days, so this is our time unit. After  $t$  days have passed, then  $\frac{t}{8}$  is the number of time units that have passed. Starting with the initial amount of 40, our half-life model becomes:

$$P(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{8}}$$

With  $t = 3$

$$P(3) = 40\left(\frac{1}{2}\right)^{\frac{3}{8}} = 30.8$$

There are 30.8 grams of the substance remaining after three days.

**Half-Life Model:** If  $H$  is the half-life of a quantity (the amount of time it takes the quantity be cut in half) and  $P_0$  is the initial amount of the quantity then the amount of the quantity present after  $t$  units of time is  $P(t) = P_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$

**Example 4.3.9: Lead-209**

Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 40 milligrams of this isotope is created in an experiment, how much is left after 14 hours? Use the half-life model to solve this problem.

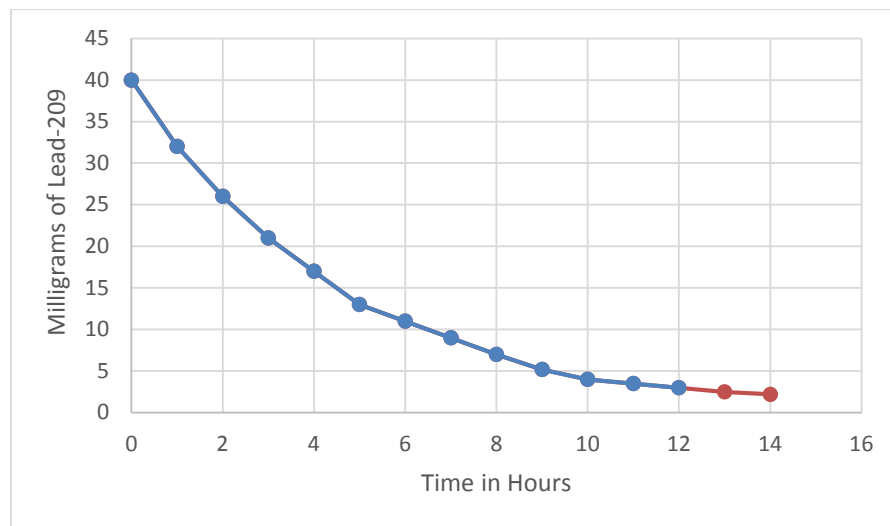
$P_0 = 40$  and  $H = 3.3$ , so the half-life model for this problem is:

$$P(t) = 40 \left( \frac{1}{2} \right)^{\frac{t}{3.3}} \text{ when } t = 14 \text{ hours,}$$

$$P(14) = 40 \left( \frac{1}{2} \right)^{\frac{14}{3.3}} = 2.1$$

There are 2.1 milligrams of Lead-209 remaining after 14 hours.

**Figure 4.3.3: Lead-209 Decay Graph**



*Note: The milligrams of Lead-209 remaining follows a decreasing exponential growth model.*

**Example 4.3.10: Nobelium-259**

Nobelium-259 has a half-life of 58 minutes. If you have 1000 grams, how much will be left in two hours? We solve this problem using the half-life model. Before we begin, it is important to note the time units. The half-life is given in minutes and we want to know how much is left in two hours. Convert hours to minutes when using the model: two hours = 120 minutes.



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$P_0 = 1000$  and  $H = 58$  minutes, so the half-life model for this problem is:

$$P(t) = 1000 \left( \frac{1}{2} \right)^{\frac{t}{58}}$$

When  $t = 120$  minutes,

$$P(120) = 1000 \left( \frac{1}{2} \right)^{\frac{120}{58}} = 238.33$$

There are 238 grams of Nobelium-259 is remaining after two hours.

### Example 4.3.11: Carbon-14

Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 1947, earthenware jars containing what are known as the Dead Sea Scrolls were found. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the age of the Dead Sea Scrolls. In this problem, we want to estimate the age of the scrolls. In 1947, 76% of the carbon-14 remained. This means that the amount remaining at time  $t$  divided by the original amount of carbon-14,

$P_0$ , is equal to 76%. So,  $\frac{P(t)}{P_0} = 0.76$  we use this fact to solve for  $t$ .

$$P(t) = P_0 \left( \frac{1}{2} \right)^{\frac{t}{5730}}$$

$$\frac{P(t)}{P_0} = \left( \frac{1}{2} \right)^{\frac{t}{5730}}$$

$$0.76 = \left( \frac{1}{2} \right)^{\frac{t}{5730}} \quad \text{Now, take the log of both sides of the equation.}$$

$$\log 0.76 = \log \left( \frac{1}{2} \right)^{\frac{t}{5730}} \quad \text{The exponent comes down using rules of logarithms.}$$

$$\log 0.76 = \left( \frac{t}{5730} \right) \log \frac{1}{2} \quad \text{Now, calculate } \log 0.76 \text{ and } \log \frac{1}{2} \text{ with your calculator.}$$

$$-0.1192 = \left( \frac{t}{5730} \right) \times (-0.3010)$$

$$\frac{-0.1192}{-0.3010} = \frac{t}{5730}$$

$$0.3960 = \frac{t}{5730}$$

$t = 2269.08$ . The Dead Sea Scrolls are well over 2000 years old.

**Example 4.3.12: Plutonium**

Plutonium has a half-life of 24,000 years. Suppose that 50 pounds of it was dumped at a nuclear waste site. How long would it take for it to decay into 10 lbs?

$P_0 = 50$  and  $H = 24,000$ , so the half-life model for this problem is:

$$P(t) = 50 \left( \frac{1}{2} \right)^{\frac{t}{24,000}}$$

Now, find  $t$  when  $P(t) = 10$ .

$$10 = 50 \left( \frac{1}{2} \right)^{\frac{t}{24,000}}$$

$$\frac{10}{50} = \left( \frac{1}{2} \right)^{\frac{t}{24,000}}$$

$$0.2 = \left( \frac{1}{2} \right)^{\frac{t}{24,000}} \quad \text{Now, take the log of both sides of the equation.}$$

$$\log 0.2 = \log \left( \frac{1}{2} \right)^{\frac{t}{24,000}} \quad \text{The exponent comes down using rules of logarithms.}$$

$$\log 0.2 = \left( \frac{t}{24,000} \right) \log \frac{1}{2} \quad \text{Now, calculate } \log 0.2 \text{ and } \log \frac{1}{2} \text{ with your calculator.}$$

$$-0.6990 = \left( \frac{t}{24,000} \right) \times (-0.3010)$$

$$\frac{-0.6990}{-0.3010} = \frac{t}{24,000}$$

$$2.322 = \frac{t}{24,000}$$

$$t = 55,728$$

The quantity of plutonium would decrease to 10 pounds in approximately 55,728 years.

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### Rule of 70 for Half-Life:

There is simple formula for approximating the half-life of a population. It is called the rule of 70 and is an approximation for decay rates less than 15%. Do not use this formula if the decay rate is 15% or greater.

**Rule of 70:** For a quantity decreasing at a constant percentage (not written as a decimal),  $R$ , per time period, the half-life is approximately given by:

$$\text{Half-life } H \approx \frac{70}{R}$$

### Example 4.3.13: Elephant Population

The population of wild elephants is decreasing by 7% per year. Approximate the half-life for this population. If there are currently 8000 elephants left in the wild, how many will remain in 25 years?

To solve this problem, use the half-life approximation formula.

$$\text{Half-life } H \approx \frac{70}{7} = 10 \text{ years}$$

$P_0 = 7000$ ,  $H = 10$  years, so the half-life model for this problem is:

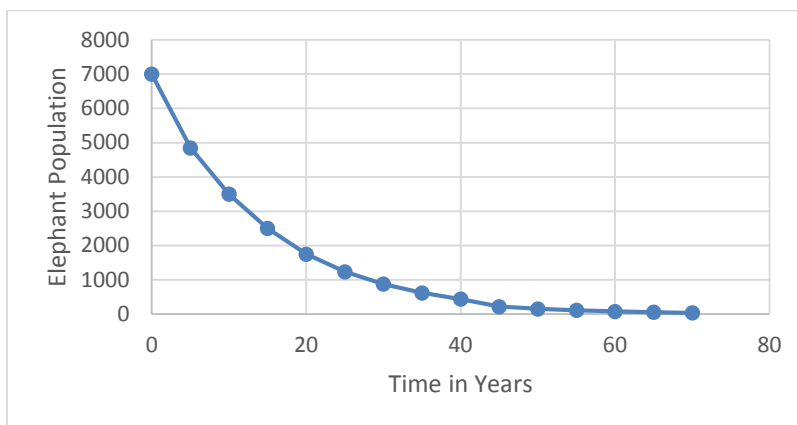
$$P(t) = 7000 \left( \frac{1}{2} \right)^{\frac{t}{10}}$$

When  $t = 25$ ,

$$P(25) = 7000 \left( \frac{1}{2} \right)^{\frac{25}{10}} = 1237.4$$

There will be approximately 1237 wild elephants left in 25 years.

**Figure 4.3.4: Elephant Population over a 70 Year Span.**



*Note: The population of elephants follows a decreasing exponential growth model.*

Review of Exponent Rules and Logarithm Rules	
Rules of Exponents	Rules of Logarithm for the Common Logarithm (Base 10)
<b>Definition of an Exponent</b> $a^n = a \cdot a \cdot a \cdot a \cdot \dots \cdot a$ ( $n$ $a$ 's multiplied together)	<b>Definition of a Logarithm</b> $10^y = x$ if and only if $\log x = y$
<b>Zero Rule</b> $a^0 = 1$	
<b>Product Rule</b> $a^m \cdot a^n = a^{m+n}$	<b>Product Rule</b> $\log(xy) = \log x + \log y$
<b>Quotient Rule</b> $\frac{a^m}{a^n} = a^{m-n}$	<b>Quotient Rule</b> $\log\left(\frac{x}{y}\right) = \log x - \log y$
<b>Power Rule</b> $(a^n)^m = a^{n \cdot m}$	<b>Power Rule</b> $\log x^r = r \log x \quad (x > 0)$
<b>Distributive Rules</b> $(ab)^n = a^n b^n, \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\log 10^x = x \log 10 = x$
<b>Negative Exponent Rules</b> $a^{-n} = \frac{1}{a^n}, \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$10^{\log x} = x \quad (x > 0)$

## Section 4.4: Natural Growth and Logistic Growth

In this chapter, we have been looking at linear and exponential growth. Another very useful tool for modeling population growth is the natural growth model. This model uses base  $e$ , an irrational number, as the base of the exponent instead of  $(1 + r)$ . You may remember learning about  $e$  in a previous class, as an exponential function and the base of the natural logarithm.

**The Natural Growth Model:**  $P(t) = P_0 e^{kt}$  where  $P_0$  is the initial population,  $k$  is the growth rate per unit of time, and  $t$  is the number of time periods.

## Chapter 4: Growth

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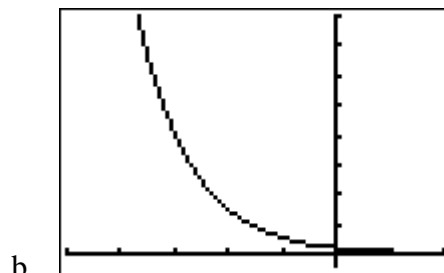
Given  $P_0 > 0$ , if  $k > 0$ , this is an exponential growth model, if  $k < 0$ , this is an exponential decay model.

**Figure 4.4.1: Natural Growth and Decay Graphs**

a. Natural growth function  $P(t) = e^t$



b. Natural decay function  $P(t) = e^{-t}$



### Example 4.4.1: Drugs in the Bloodstream

When a certain drug is administered to a patient, the number of milligrams remaining in the bloodstream after  $t$  hours is given by the model  $P(t) = 40e^{-.25t}$

How many milligrams are in the blood after two hours?

To solve this problem, we use the given equation with  $t = 2$

$$P(2) = 40e^{-.25(2)}$$

$$P(2) = 24.26$$

There are approximately 24.6 milligrams of the drug in the patient's bloodstream after two hours.

In the next example, we can see that the exponential growth model does not reflect an accurate picture of population growth for natural populations.

### Example 4.4.2: Ants in the Yard

Bob has an ant problem. On the first day of May, Bob discovers he has a small red ant hill in his back yard, with a population of about 100 ants. If conditions are just right red ant colonies have a growth rate of 240% per year during the first four years. If Bob does nothing, how many ants will he have next May? How many in five years?

We solve this problem using the natural growth model.

$$P(t) = 100e^{2.4t}$$

In one year,  $t = 1$ , we have

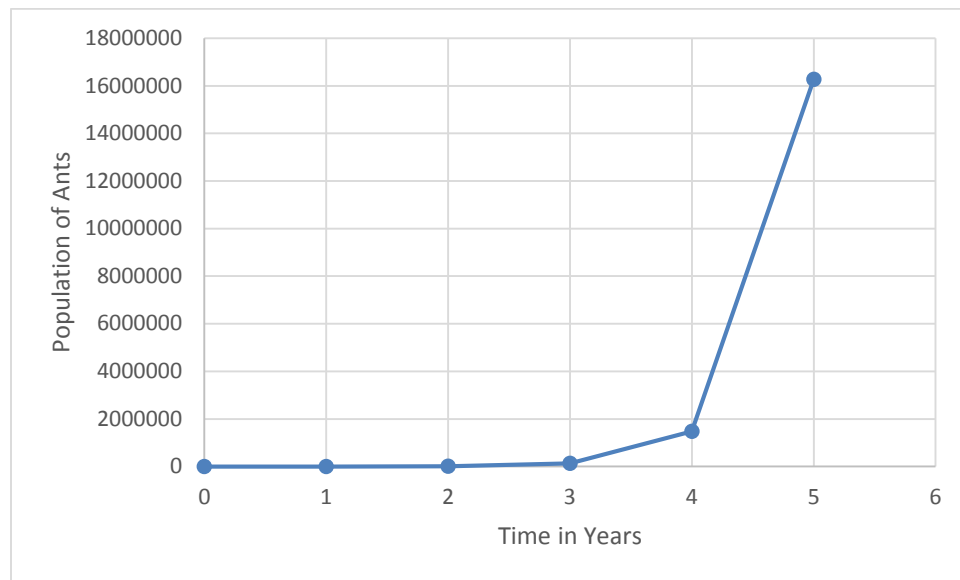
$$P(1) = 100e^{2.4(1)} = 1102 \text{ ants}$$

In five years,  $t = 5$ , we have

$$P(5) = 100e^{2.4(5)} = 16,275,479 \text{ ants}$$

That is a lot of ants! Bob will not let this happen in his back yard!

**Figure 4.4.2: Graph of Ant Population Growth in Bob's Yard.**



*Note: The population of ants in Bob's back yard follows an exponential (or natural) growth model.*

The problem with exponential growth is that the population grows without bound and, at some point, the model will no longer predict what is actually happening since the amount of resources available is limited. Populations cannot continue to grow on a purely physical level, eventually death occurs and a limiting population is reached.

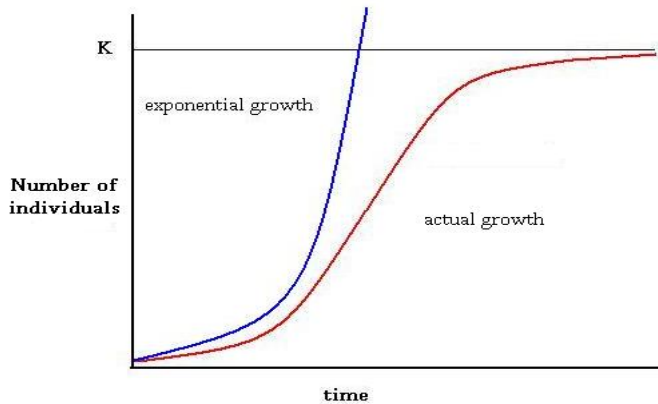
Another growth model for living organisms is the logistic growth model. The logistic growth model has a maximum population called the carrying capacity. As the population grows, the number of individuals in the population grows to the carrying capacity and stays there. This is the maximum population the environment can sustain.

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**Logistic Growth Model:**  $P(t) = \frac{M}{1 + ke^{-ct}}$  where  $M$ ,  $c$ , and  $k$  are positive constants and  $t$  is the number of time periods.

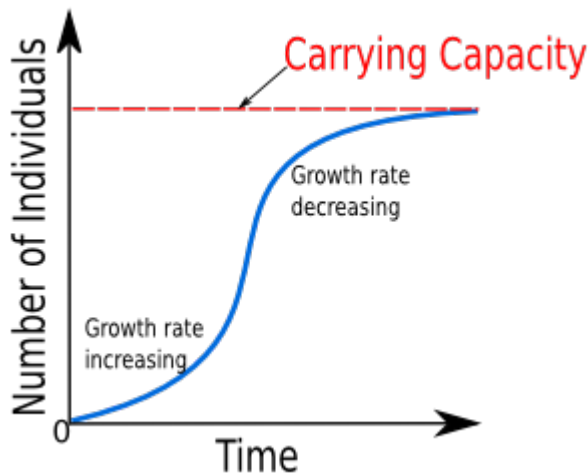
### Figure 4.4.3: Comparison of Exponential Growth and Logistic Growth

The horizontal line  $K$  on this graph illustrates the carrying capacity. However, this book uses  $M$  to represent the carrying capacity rather than  $K$ .



(Logistic Growth Image 1, n.d.)

### Figure 4.4.4: Logistic Growth Model



(Logistic Growth Image 2, n.d.)

The graph for logistic growth starts with a small population. When the population is small, the growth is fast because there is more elbow room in the environment. As the population approaches the carrying capacity, the growth slows.

**Example 4.4.3: Bird Population**

The population of an endangered bird species on an island grows according to the logistic growth model.

$$P(t) = \frac{3640}{1 + 25e^{-0.04t}}$$

Identify the initial population. What will be the bird population in five years? What will be the population in 150 years? What will be the population in 500 years?

We know the initial population,  $P_0$ , occurs when  $t = 0$ .

$$P_0 = P(0) = \frac{3640}{1 + 25e^{-0.04(0)}} = \frac{3640}{1 + 25e^{-0.04(0)}} = 140$$

Calculate the population in five years, when  $t = 5$ .

$$P(5) = \frac{3640}{1 + 25e^{-0.04(5)}} = \frac{3640}{1 + 25e^{-0.04(5)}} = 169.6$$

The island will be home to approximately 170 birds in five years.

Calculate the population in 150 years, when  $t = 150$ .

$$P(150) = \frac{3640}{1 + 25e^{-0.04(150)}} = \frac{3640}{1 + 25e^{-0.04(150)}} = 3427.6$$

The island will be home to approximately 3428 birds in 150 years.

Calculate the population in 500 years, when  $t = 500$ .

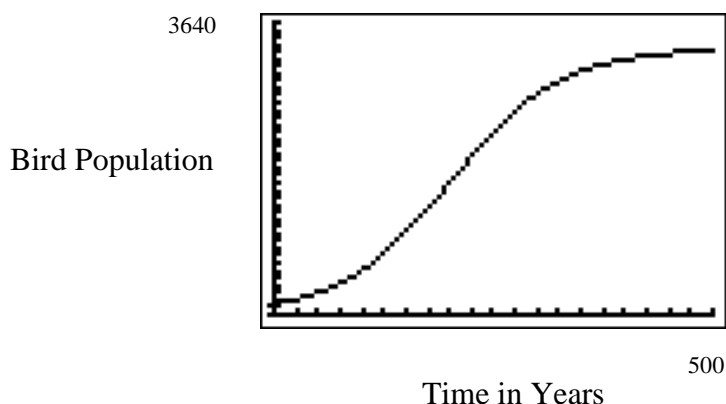
$$P(500) = \frac{3640}{1 + 25e^{-0.04(500)}} = \frac{3640}{1 + 25e^{-0.04(500)}} = 3640.0$$

The island will be home to approximately 3640 birds in 500 years.

This example shows that the population grows quickly between five years and 150 years, with an overall increase of over 3000 birds; but, slows dramatically between 150 years and 500 years (a longer span of time) with an increase of just over 200 birds.



**Figure 4.4.5: Bird Population over a 200-Year Span**



### Example 4.4.4: Student Population at Northern Arizona University

The student population at NAU can be modeled by the logistic growth model below, with initial population taken from the early 1960's. We will use 1960 as the initial population date.

$$P(t) = \frac{30,000}{1 + 5e^{-0.06t}}$$

Determine the initial population and find the population of NAU in 2014. What will be NAU's population in 2050? From this model, what do you think is the carrying capacity of NAU?

We solve this problem by substituting in different values of time.

When  $t = 0$ , we get the initial population  $P_0$ .

$$P_0 = P(0) = \frac{30,000}{1 + 5e^{-0.06(0)}} = \frac{30,000}{6} = 5000$$

The initial population of NAU in 1960 was 5000 students.

In the year 2014, 54 years have elapsed so,  $t = 54$ .

$$P(54) = \frac{30,000}{1 + 5e^{-0.06(54)}} = \frac{30,000}{1 + 5e^{-3.24}} = \frac{30,000}{1.19582} = 25,087$$

There are 25,087 NAU students in 2014.

In 2050, 90 years has elapsed so,  $t = 90$ .

$$P(90) = \frac{30,000}{1 + 5e^{-0.06(90)}} = \frac{30,000}{1 + 5e^{-5.4}} = 29,337$$

There are 29,337 NAU students in 2050.

Finally, to predict the carrying capacity, look at the population 200 years from 1960, when  $t = 200$ .

$$P(200) = \frac{30,000}{1 + 5e^{-0.06(200)}} = \frac{30,000}{1 + 5e^{-12}} = \frac{30,000}{1.00003} = 29,999$$

Thus, the carrying capacity of NAU is 30,000 students.

It appears that the numerator of the logistic growth model,  $M$ , is the carrying capacity.

**Carrying Capacity:** Given the logistic growth model  $P(t) = \frac{M}{1 + ke^{-ct}}$ , the carrying capacity of the population is  $M$ .  $M$ , the carrying capacity, is the maximum population possible within a certain habitat.

#### Example 4.4.5: Fish Population

Suppose that in a certain fish hatchery, the fish population is modeled by the logistic growth model where  $t$  is measured in years.

$$P(t) = \frac{12,000}{1 + 11e^{-0.2t}}$$

What is the carrying capacity of the fish hatchery? How long will it take for the population to reach 6000 fish?

The carrying capacity of the fish hatchery is  $M = 12,000$  fish.

Now, we need to find the number of years it takes for the hatchery to reach a population of 6000 fish. We must solve for  $t$  when  $P(t) = 6000$ .

$$6000 = \frac{12,000}{1 + 11e^{-0.2t}}$$

$$(1 + 11e^{-0.2t}) \cdot 6000 = \frac{12,000}{1 + 11e^{-0.2t}} \cdot (1 + 11e^{-0.2t})$$

$$(1 + 11e^{-0.2t}) 6000 = 12,000$$

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$$\frac{(1+11e^{-0.2t})\cancel{6000}}{\cancel{6000}} = \frac{12,000}{6000}$$

$$1+11e^{-0.2t} = 2$$

$$11e^{-0.2t} = 1$$

$e^{-0.2t} = \frac{1}{11} = 0.090909$  Take the natural logarithm (ln on the calculator) of both sides of the equation.

$$\ln e^{-0.2t} = \ln 0.090909 \quad \ln e^{-0.2t} = -0.2t \text{ by the rules of logarithms.}$$

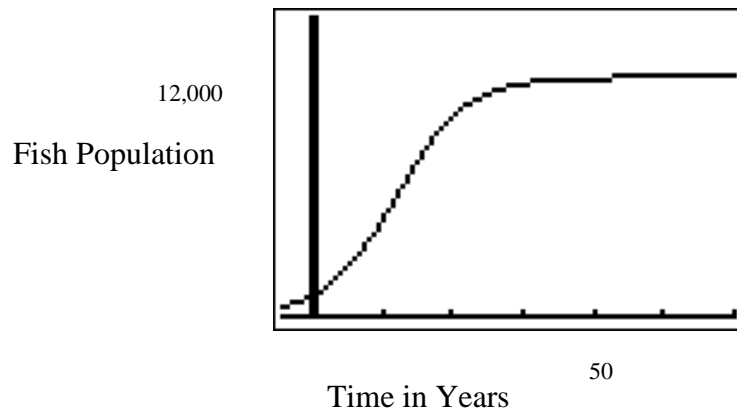
$$-0.2t = \ln 0.090909$$

$$t = \frac{\ln 0.090909}{-0.2}$$

$$t = 11.999$$

It will take approximately 12 years for the hatchery to reach 6000 fish.

**Figure 4.4.6: Fish Population over a 30-Year Period.**



## Chapter 4 Homework

1. Goodyear AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 72,800. The city's population grew by 3800 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Goodyear. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)
2. Gilbert AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 229,800. The city's population grew by 9200 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Gilbert. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)
3. Apache County AZ, is shrinking according to the census bureau. In 2012, the population was about 71,700. The county's population decreased by 1147 people from 2012 to 2013. If the decline is linear, create a population model for Apache County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)
4. Cochise County AZ, is shrinking according to the census bureau. In 2012, the population was about 129,472. The county's population decreased by 2600 people from 2012 to 2013. If the decline is linear, create a population model for Cochise County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)
5. Mohave County AZ, is shrinking according to the census bureau. In 2012, the population was about 203,030. The county's population decreased by 1200 people from 2012 to 2013. If the decline is linear, create a population model for Mohave County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)

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6. The 2012 Kia Sedona LX has one of the largest depreciation values of any car. Suppose a 2012 Kia Sedona sold for \$24,900, and its value depreciates by \$3400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
  
7. The 2013 Chevy Impala has one of the largest depreciation values of any car. Suppose a 2013 Chevy Impala sold for \$27,800 and its value depreciates by \$3600 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
  
8. The 2013 Jaguar XJ AWD has one of the largest depreciation values of any car. Suppose a 2013 Jaguar XJ AWD sold for \$74,500 and its value depreciates by \$10,400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
  
9. The 2012 Jeep Liberty Limited Sport 2WD has one of the largest depreciation values of any car. Suppose a 2012 Jeep Liberty Limited Sport 2WD sold for \$23,400 and its value depreciates by \$3,040 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)
  
10. Suppose that in January, the maximum water depth at Lake Powell, Arizona was 528 feet. The water evaporates at an average rate of 1.2 feet per month. Find a model for the rate at which the water evaporates. If it does not rain at all, what will be the depth of Lake Powell in May and in September?
  
11. Suppose that the maximum water depth at Lake Tahoe, California in 2014 was 1644 feet. Because of the drought, the water level has been decreasing at an average rate of 6.2 feet per year. Find a model for the rate at which the water level decreases. If there is no precipitation at all, what will be the depth of Lake Tahoe be in two years, and in five years?

12. Suppose the homes in Arizona have appreciated an average of 8% per year in the last five years. If the average home in a suburb sold for \$225,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?
  
13. Suppose the homes in Massachusetts have appreciated an average of 13% per year in the last five years. If the average home in a suburb sold for \$205,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?
  
14. Suppose the homes in Michigan have depreciated an average of 17% per year in the last five years. If the average home in a suburb sold for \$215,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?
  
15. Suppose the homes in Nevada have depreciated an average of 15% per year in the last five years. If the average home in a suburb sold for \$318,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?
  
16. The cost of a home in Flagstaff AZ was \$89,000 in 1992. In 2007, the same home appraised for \$349,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?
  
17. The cost of a home in Bullhead City AZ was \$109,000 in 1992. In 2007, the same home appraised for \$352,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?

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18. The population of West Virginia is in decline. The population in 2014 was 1,850,326 and the population had decreased by 0.14% from 2010. How many people were living in West Virginia in 2010? Create a model for this population. If the decline continues at this rate, how many people will reside in West Virginia in 2020? (Wikipedia, n.d.)
  
19. Assume that the population of Arizona grew by 2.4% per year between the years 2000 to 2010. The number of Native American living in Arizona was 257,426 in 2010. How many Native Americans were living in Arizona in 2000? Create a model for this population. If the growth continues at this rate, how many Native Americans will reside in Arizona in 2020?
  
20. Assume the population of the U.S. grew by 0.96% per year between the years 2000 and 2010. The number of Hispanic Americans was 55,740,000 in 2010. How many Hispanic Americans were living in the U.S. in 2000? Create a model for this population. If the growth continues at this rate, how many Hispanic Americans will reside in The U.S. in 2020?
  
21. Assume the population of Michigan decreased by 0.6% per year between the years 2000 to 2010. The population of Michigan was 9,970,000 in 2010. How many people were living in Michigan in 2000? Create a model for this population. If the growth continues at this rate, how many people will reside in Michigan in 2020?
  
22. The doubling time of a population of aphids is 12 days. If there are initially 200 aphids, how many aphids will there be in 17 days?
  
23. The doubling time of a population of rabbits is six months. If there are initially 26 rabbits, how many rabbits will there be in 17 months?
  
24. The doubling time of a population of shrews is three months. If there are initially 32 shrews, how many shrews will there be in 21 months?

25. The doubling time of a population of hamsters is 1.2 years. If there are initially 43 hamsters, how many hamsters will there be in 7 years?
26. A certain cancerous tumor doubles in size in three months. If the initial size of the tumor is two cells, how many months will it take for the tumor to grow to 60,000 cells? How many cells will there be in 1.5 years? In three years?
27. A certain cancerous tumor doubles in size in 1.5 months. If the initial size of the tumor is eight cells, how many months will it take for the tumor to grow to 40,000 cells? How many cells will there be in six months? In 2.5 years?
28. A bird population on a certain island has an annual growth rate of 1.5% per year. Approximate the number of years it will take the population to double. If the initial population is 130 birds, use it to find the bird population of the island in 14 years.
29. The beaver population on Kodiak Island has an annual growth rate of 1.2% per year. Approximate the number of years it will take the population to double. If the initial population is 32 beavers, use it to find the population of beavers on the island in 20 years.
30. The black-footed ferret population in Arizona has an annual growth rate of 0.5% per year. Approximate the number of years it will take the population to double. If the initial population is 230 ferrets, use it to find the ferret population in AZ in 12 years.
31. The Mexican gray wolf population in southern Arizona increased from 72 individuals to 92 individuals from 2012 to 2013. What is the annual growth rate? Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Mexican gray wolves in 10 years.



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32. There is a small population of Sonoran pronghorn antelope in a captive breeding program on the Cabeza Prieta National Wildlife Refuge in southwest Arizona. Recently, the population increased from 122 individuals to 135 individuals from 2012 to 2013. Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Sonoran pronghorn antelope in 10 years.
33. Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 56 milligrams of this isotope is created in an experiment, how much is left after 12 hours?
34. Titanium-44 has a half-life of 63 years. If there is 560 grams of this isotope, how much is left after 1200 years?
35. Uranium-232 has a half-life of 68.9 years. If there is 160 grams of this isotope, how much is left after 1000 years?
36. Nickel-63 has a half-life of 100 years. If there is 16 grams of this isotope, how much is left after 145 years?
37. Radium-226 has a half-life of 1600 years. If there is 56 grams of this isotope, how much is left after 145,000 years?
38. The population of wild desert tortoises is decreasing by 72% per year. Approximate the half-life for this population. If there are currently 100,000 tortoises left in the wild, how many will remain in 20 years?
39. The population of pygmy owls is decreasing by 4.4% per year. Approximate the half-life for this population. If there are currently 41 owls left in the wild, how many will remain in five years?

40. Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 2004, carbonized plant remains were found where human artifacts were unearthed along the Savannah River in Allendale County. Analysis indicated that the plant remains contained 0.2% of their original carbon-14. Estimate the age of the plant remains. (Wikipedia, n.d.)

41. The population of a bird species on an island grows according to the logistic model below.

$$P(t) = \frac{43,240}{1 + 12e^{-0.05t}}$$

Identify the initial population. What will be the bird population in five years? In 150 years? In 500 years? What is the carrying capacity of the bird population on the island?

42. Suppose the population of students at Arizona State University grows according to the logistic model below with the initial population taken from 2009.

$$P(t) = \frac{85,240}{1 + 0.6e^{-0.15t}}$$

Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at ASU?

43. Suppose the population of students at Ohio State University grows according to the logistic model below with the initial population taken from 1970.

$$P(t) = \frac{90,000}{1 + 4e^{-0.06t}}$$

Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at OSU?

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44. Suppose that in a certain shrimp farm, the shrimp population is modeled by the logistic model below where  $t$  is measured in years.

$$P(t) = \frac{120,000}{1 + 25e^{-0.12t}}$$

Find the initial population. How long will it take for the population to reach 6000 shrimp? What is the carrying capacity of shrimp on the farm?

45. Suppose that in a certain oyster farm, the oyster population is modeled by the logistic model below where  $t$  is measured in years.

$$P(t) = \frac{18,000}{1 + 19e^{-0.36t}}$$

Find the initial population. How long will it take for the population to reach 2000 oysters? What is the carrying capacity of oysters on the farm?